Upgrades for Teaching CE En/ME En 506

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ABSTRACT

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This paper describes a project concerned mainly with improving teaching and learning in the course CE En/ME En 506 Continuum Mechanics and Finite Elements taught by Dr. Michael A. Scott of the BYU Civil and Environmental Engineering department. The project is aimed at developing a course in which students are able to succeed and an environment that will stimulate student interest in research into the field of computational mechanics. In pursuing this goal, the effort put into teaching the class and the methods used to teach the subject will play a significant role. The paper describes some of the improvements that have already been made over the course of the two years the class has been taught. The work done for the project mainly consisted of design and introduction of new homework assignments and a new coding project in which, for the first time, the students write code that works in multiple dimensions. So far, the new homework assignments will, in the future, necessitate enhanced teaching of mathematical manipulation techniques since they involve more complex computations than the students seemed prepared for this semester. The coding project has been a success with the only problems resulting from a significant increase in completion time for the students over previous semester's coding assignments. Continued work on further enhancement of the upgrades introduced this semester ought to produce noticeable progress in the students interest and abilities in finite elements.

Keywords: finite element method, finite element analysis, computational mechanics, continuum mechanics, isogeometric analysis

ACKNOWLEDGMENTS

Though there have been many who've made significant contributions to my life, I would especially like to acknowledge the wonderful support and help that my family has been to me. My older brothers paved the way by establishing successful lives of their own and provided good examples for me to model in my own life. My father and mother have always sought to instill in me that I had the capability to succeed no matter the obstacle, and no matter the failure I always had the ability improve.

I also feel it important, as a believing Latter-day Saint, to also acknowledge the assistance of a loving Father in Heaven. I have found a consistent source of hope in the life of His son the Lord Jesus Christ and an ever-present means to personal inspiration through the Holy Ghost. The belief I have in God has provided the foundation for any success I've been able to reach in life.

It is anticipated that the completion of this project will bring an end to my formal education. As with all endings, it is natural to look back at the challenges and successes, the hard times and the good times, and to self-evaluate how I've done over my many years as a student. Overall I consider my education a wild success and it is something for which I am truly grateful. I am also careful not to take too much credit for the successes that could be seen as mine alone. It has been said that it takes a village to raise a child, and to that I would add that it takes a city to raise a graduate student. As a student, I've had the opportunity to be the recipient of so much assistance in the form of counseling, teaching, and mentorship from many different people who have at the same time pushed, motivated, and inspired me toward fulfillment of my own potential. This has set me up well to move on from school into a life where there are many opportunities available to me and which I am incredibly excited to begin.

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CHAPTER 1. INTRODUCTION

This paper will summarize the work I have done in cooperation with Dr. Michael Scott within the Department of Civil and Environmental Engineering for the design of upgrades to the curriculum for the course CE En/ME En 506 Continuum Mechanics and Finite Elements (hereafter referred to as CE 506). The goal is to create an academically challenging and rigorous environment for the study of computational mechanics at Brigham Young University and to place the civil engineering department's computational mechanics courses and research activities on the cutting edge of the field. Currently the department's faculty already possesses the expertise necessary to produce such research; for an example see [2]. However, the difficulty has been in creating student interest in the subject and transitioning them from their undergraduate coursework to a level where they can succeed in related research activities. In order to overcome this difficulty, the effort devoted to teaching this class is of significant importance. Therefore, this project has been primarily concerned with the design of homework assignments and coding projects that will create a course with the potential to help realize this goal. These assignments will require students to perform much more advanced computations than has previously been attempted in the course and will be taught in a way that allows students' understanding to progress at a pace that will help them succeed.

1.1 Computational Mechanics at BYU

Computational mechanics seeks to apply methods from numerical analysis and computer science to facilitate better solutions to physical problems governed by traditional continuum mechanics. The rapid advances seen in the computing industry have allowed researchers in this field to solve bigger and more complex problems at an aggressive rate. Thus, this field of research is expanding and has broad implications for advances in science and engineering. In 2006, a report released by the National Science Foundation discussed the need for and benefits of increased national attention and investment in a field they define as simulation-based engineering science (SBES), of which computational mechanics is a subset. The report states, "In fact, so profound are the implications of advanced simulation techniques that we can expect SBES to trigger the development of a host of aggressive new technologies and to foster significant new scientific discoveries". The report goes on, "Indeed, seldom have so many independent studies by experts from diverse perspectives been in such agreement: computer simulation has and will continue to have an enormous impact on all areas of engineering, scientific discovery, and endeavors to solve major societal problems" [3].

Within the framework of simulation-based engineering and science, the finite element method often becomes the workhorse for obtaining solutions to complex problems thousands of times over. Therefore, simulation technology can be highly dependent on the efficiency and accuracy of the methods by which these solutions are obtained. The research done at BYU targets this area directly with Dr. Scott's focus on research into isogeometric analysis. Dr. Thomas J.R. Hughes first introduced isogeometric analysis in 2005 [4] as a potential alternative or update to traditional finite element methods. Since its introduction, the technology has been adopted as the subject of considerable research by many institutions worldwide. Fundamental to a student's ability to effectively participate in research into isogeometric analysis is a solid understanding of the principals of the finite element method. CE 506 is the first introduction students in the civil engineering program receive to the finite element method. If they are to eventually do research in this area, their first introduction should be taught in a way that opens the door to interest in participating in this growing field.

1.2 Good teaching

Much of the reason for the effort devoted to this project has come because Dr. Scott has observed that students tend to consider this class very difficult, and they often report not feeling prepared to understand the mathematical and programming concepts taught in the class. This is a situation in which the professor needs to devote serious effort to teaching a class well. Without good teaching the problems observed by both Dr. Scott and his students will likely continue, resulting in a class providing limited exposure to finite element analysis. Consequently, students may not fully realize their potential as researchers with such a small introduction to the subject.

Though a student's ability to succeed as a researcher will not solely depend on the quality of this class, the course is an important step in their abilities and is a teaching opportunity that should not be taken lightly.

Every professor and every department dreams of teaching, mentoring, and graduating students who are self motivated, who are excited about the things they learn in class, and who want to participate in research. Often though, we find that students seem more concerned about deadlines, grades, and workloads than learning. In my 20 years experience as a "professional" student I have noticed that I have played both roles at one time or another. Whether I was the student interested in learning or whether I seemed the student who just wanted to get the class over with depended quite a bit on the teaching I received. In addition, consider some responses from other students from a variety of educational backgrounds to an informal survey I conducted. I asked the students to think of one or two teachers they've had at some point in their education who they considered talented teachers and explain how these teachers' teaching abilities/methods affected them as students. I have included some of their responses below along with their majors:

- "The main thing is that they helped me think the subject was cool or interesting, whereas if they weren't a good teacher, I would have thought the subject lame or boring. Things like computational theory or math come to mind. Also, a good teacher made it so that even if I struggled to get a good grade, I still enjoyed learning the concepts and lectures."
  –Computer Science, Graduate Student

- "He expected more of us than almost any teacher in the school (my high school's graduation rate is under 60%). We actually had to read the textbook and take good notes to do well... The effort he put into lessons and discussions showed us he cared, and that made us push harder... He expected more of us, so we gave him more."
  –Social Science Teaching

- "The main effect this had on me was inspiring a personal interest in the subject. The result of this personal interest is a permanent knowledge of the subject."
  –Mechanical Engineering, Graduate Student

- "She made every lesson seem fascinating, because SHE thought Portuguese grammar was fascinating." –International Relations

- "I think something interesting about my favorite teachers is that many of them are from my hardest classes and I didn't do especially well in them. They set high expectations but are also aware of limitations and are willing to work with you. Because of that, you learn more than just the material but how to problem solve and be diligent."
  –Illustration

- "I was more excited about learning and participating in class, I felt obligated to do better in their classes, and I talked more about what I was learning in class outside of class."
  –Medical Student

- "[It] makes me as a student care a lot more about class and what I'm contributing. I feel like I enjoy the classes more, get more out of lectures, try harder on the readings, etc."
  –Law Student

- "The real question to evaluate teachers on this point is: Do the students feel comfortable asking questions and occasionally looking stupid? If not, the teacher may be brilliant, but ultimately unable to transfer knowledge."
  –Mechanical Engineering

- "I wanted to go to class because I knew the lectures would be great and I knew I would walk away from that hour feeling like it was actually worth going to class. Then when I got home I would want to study the material and do the homework... It didn't just feel like a check off the list towards a career."
  –Dental Student

- "I found that I really enjoyed learning about it and I think because of his enthusiasm. He made the material seem very relevant and the things he taught still linger in my mind."
  –Biomedical Engineering, Graduate Student

Note that the survey did not ask for ideas that professors can use to improve their teaching but simply to explain how teachers who tried hard to be good teachers affected them. The reason for this is that since professor personalities and course subjects vary so widely, I wished to emphasize only the effect and benefits of good teaching as identified by a diverse group of respondents. Judging from personal interactions I've had with professors at BYU, I have determined that each possesses the ability, if they so desire, to learn to be great teachers without suggestions from this paper. However, in gathering these responses, the students naturally tended to offer their own suggestions and observations on what makes a good teacher; the interested reader may view all responses in full in section A.1 of the appendix.

The responses I have obtained give an indication that students are responsive to teachers' efforts to be good teachers. In these responses we see that a teacher can help students find interests they never knew they had, push them to try harder, help them emphasize learning over deadlines, and even cause them to appreciate classes they don't get good grades in! For any professor wondering how to create real interest in research, how to boost student ratings of their courses, or how to make their classes a better learning environment, I hope these responses provide motivation to devote significant effort to developing good teaching skills. In writing this paper, I have assumed that most professors choose their career path out of a desire, at least in part, to help students succeed in their own lives and careers. With that in mind, the responses I've gathered should also provide some validation that the work a professor puts into their teaching activities is meaningful and appreciated by their students. Applied directly to the task addressed by this project, the responses here also offer hope that efforts devoted to the improved teaching of CE 506 will pay off. As teaching of CE 506 improves, we will find students motivated to work harder, to learn the new concepts that come with the class, to be successful in the assignments given, to engage the professor with questions when concepts are unclear, and to continue to develop skills necessary to pursue research into the subject, potentially with Dr. Scott.

CHAPTER 2. PROGRESS SO FAR

Dr. Scott has taught CE 506 in its entirety twice since first beginning in January 2013 and is currently finishing his third teaching of the course. Dr. Scott's approach to teaching the class represents a departure from the way in which the class has historically been taught. Rather than using a customized textbook created by BYU faculty, Dr. Scott introduced the idea of using a text [5] written by Dr. Thomas J.R. Hughes of the Institute for Computational Engineering and Sciences at the University of Texas at Austin. As alluded to previously, Dr. Hughes is a pioneer in computational mechanics and even among experts is considered one of the world's foremost authorities in computational mechanics research. Also, Dr. Scott has placed strong emphasis on successful completion of several assignments in which the students implement their own legitimate finite element code to solve several simple problems. Though the problems the students solve may in and of themselves be simple problems, success in solving the problems requires a clear understanding of how to implement the basic mathematics underpinning the finite element method.

2.1 CE 506 Winter, Fall, 2013

The culminating coding projects in the first two teachings of CE 506 centered around solving a one-dimensional beam deflection problem for various mesh sizes. In the textbook the strong form of the problem is set up in this manner, [5] where the function u is the displacement function for the beam:

Given $f : \bar{\Omega} \rightarrow \mathbb{R}$ and constants g and h, find $u : \bar{\Omega} \rightarrow \mathbb{R}$, such that:

$$\begin{aligned} u_{,xx} + f &= 0 \qquad \text{on } \Omega \\ u(1) &= g \\ -u_{,x}(0) &= h \end{aligned}$$

In the first teaching (Winter Semester 2013), the final project required the students to write a code that would solve this problem with a linear, quadratic, and cubic Lagrange polynomial basis. This project allowed the students to see how convergence rates and the accuracy of approximated solutions depend on the degree of the basis and the size of the mesh, and they were able to employ the same type of basis as is commonly used in many commercial finite element codes. The second semester's teaching introduced a new culminating coding assignment which would result in a code that solved the same problem as before, but the students would replace the Lagrange basis with a B-spline basis incorporated into their code via Bezier extraction. Figure 2.1 is a plotted example typical of the output of the code for the projects completed both semesters. The plotted functions represent the exact and approximate solutions to the beam deflection problem.

The progress made in just these two semesters does in fact represent significant progress in the teaching of finite elements at BYU. Over the course of two teachings of the class, students have begun to fully apply finite element concepts to produce their own code that can solve problems. Furthermore, with the addition of a B-spline basis via Bezier extraction, the students have progressed even beyond traditional finite element methods into the realm of isogeometric analysis. This is a step from the teaching of finite element concepts that have existed since the 1960s to concepts that are on the forefront of contemporary computational mechanics research. The nature of the projects is such that a student's code simply will not work unless each concept taught in class is thoroughly understood, so that they can apply it correctly. The skills they learn in implementing principals from class, debugging code, and explaining the results of their code are essential to being able to progress into a research environment in computational mechanics if they so desire.

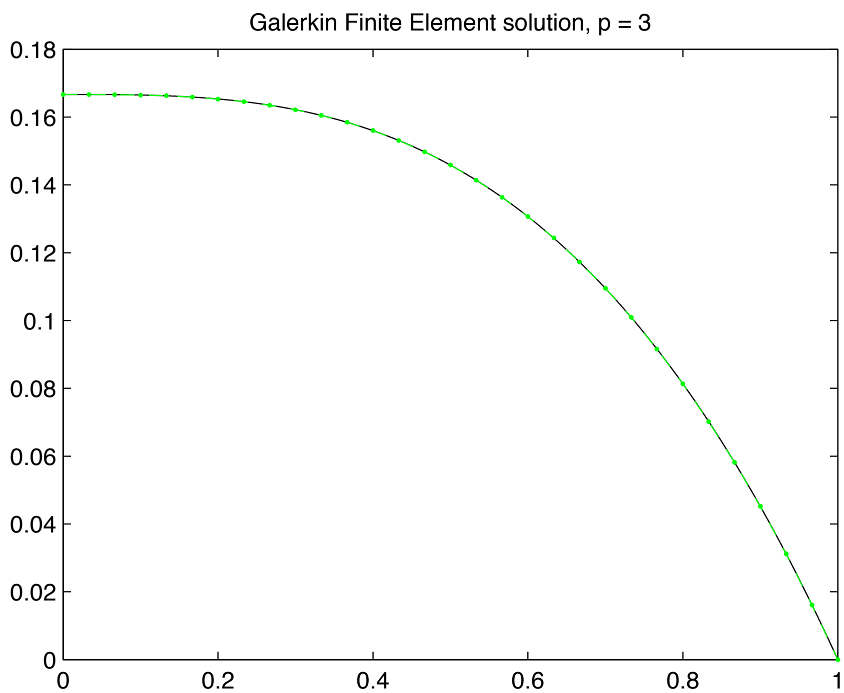

Figure 2.1: Typical solution plot for the coding projects completed in CE 506 Winter and Fall 2013

CHAPTER 3. THE PROJECT

During the course of this semester, the work I have done for this project has manifested itself in acting as a TA for CE 506. In doing so I have attended class lectures, held review sessions, worked individually with students, and collaborated on assignments and teaching strategies with Dr. Scott. By attending class I was able to stay fresh on the material and keep myself current with what the students were being taught. The review sessions provided a less formal setting for teaching, asking questions, and student collaboration than typically occurs in a classroom. Individualized help provided a chance for a student to clarify concepts that may have been understood by the majority of the other students but were still causing confusion for that student. In collaboration with Dr. Scott on assignments and teaching strategies, my connection with the students and my perspective as a student myself helped suggest approaches to the class that would unify the goals of the class with the progress of the students.

More specifically, in fulfillment of this project, Dr. Scott and I have designed two new homework assignments, a new two-dimensional finite element coding project, and a new approach to teaching the skills necessary for the coding project. The computations the students have accomplished in the class this semester have been more involved and more advanced than what has previously been done in any teaching of CE 506 at BYU. This semester therefore represents a significant step in creating a rigorous introductory course to computational mechanics that can be sufficient to set students on a course for success in graduate level research on the subject.

3.1 Homework Assignments

The first new homework assignment (HW 4) draws heavily from topics that Dr. Scott taught while I was a student in his CE En 608 class. CE 608 is meant to allow students to continue their study of finite elements and to add principals that will prepare them to solve non-linear problems. During the semester I was in the class, much of the beginning is spent introducing students to the manipulations and hand calculation style common in computational and continuum mechanics. These skills had been covered to some extent in CE 506 but not to the degree that students become comfortable with them. Dr. Scott and I determined that it might be a good approach to spend some time in CE 506 having the students become accustomed to these types of computations in order to better prepare them for CE 608 (or whatever this class will be called in the future). Therefore, in HW 4 topics such as vector projection, change of basis, index notation, and tensor calculus are introduced. The entire assignment and solution is included in section A.2 of the appendix and a sample problem from the assignment and its solution follows below.

HW 4, Problem 5:

Consider the scalar field $\varphi(\mathbf{x}) = (x_1)^2 x_3 + x_2 (x_3)^2$ and the vector field $\mathbf{v(x)} = x_3\mathbf{e}_1 + x_2 \sin(x_1)\mathbf{e}_3$. Find the components of $\triangledown \varphi(\mathbf{x})$ and $\triangledown \mathbf{v(x)}$

Solution:

The gradient of the scalar field:

$$\begin{aligned}
\varphi(\mathbf{x}) &= (x_1)^2 x_3 + x_2 (x_3)^2 \\
\triangledown \varphi(\mathbf{x}) &= \frac{\partial \varphi}{\partial x_1}\mathbf{e}_1 + \frac{\partial \varphi}{\partial x_2}\mathbf{e}_2 + \frac{\partial \varphi}{\partial x_3}\mathbf{e}_3 \\
&= (2x_1x_3)\mathbf{e}_1 + (x_3^2)\mathbf{e}_2 + (x_1^2 + 2x_2x_3)\mathbf{e}_3 \\
&= (2x_1x_3, x_3^2, x_1^2 + 2x_2x_3)
\end{aligned}$$

The gradient of the vector field:

$$
\begin{aligned}
\mathbf{v}(\mathbf{x}) &= x_3\mathbf{e}_1 + x_2\sin(x_1)\mathbf{e}_3 \\
\bigtriangledown(\mathbf{x}) &= \frac{\partial v_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j \\
&= \frac{\partial v_1}{\partial x_1}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\partial v_2}{\partial x_1}\mathbf{e}_2 \otimes \mathbf{e}_1 + \frac{\partial v_3}{\partial x_1}\mathbf{e}_3 \otimes \mathbf{e}_1 + \frac{\partial v_1}{\partial x_2}\mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{\partial v_2}{\partial x_2}\mathbf{e}_2 \otimes \mathbf{e}_2 + \ldots \\
&\quad \ldots \frac{\partial v_3}{\partial x_2}\mathbf{e}_3 \otimes \mathbf{e}_2 + \frac{\partial v_1}{\partial x_3}\mathbf{e}_1 \otimes \mathbf{e}_3 + \frac{\partial v_2}{\partial x_3}\mathbf{e}_2 \otimes \mathbf{e}_3 + \frac{\partial v_3}{\partial x_3}\mathbf{e}_3 \otimes \mathbf{e}_3 \\
&= 0 + 0 + x_2\cos x_1\mathbf{e}_3 \otimes \mathbf{e}_1 + 0 + 0 + \sin(x_1)\mathbf{e}_3 \otimes \mathbf{e}_2 + (1)\mathbf{e}_1 \otimes \mathbf{e}_3 + 0 + 0 \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ x_2\cos(x_1) & \sin x_1 & 0 \end{bmatrix}
\end{aligned}
$$

The next homework assignment was assigned as the fifth assignment in CE 506 (HW 5) and also draws heavily from topics covered in CE 608, employing the same rationale as for HW 4. HW 5 continues to focus on tensor calculus and index notation concepts with applications to stress tensors. In addition, the assignment gives the students an opportunity to derive Bernoulli's fluid flow equations from first principals. These equations play an important role in classes that most civil engineering students will have already taken as undergraduates. With this assignment, students will be able to draw connections from the new, unfamiliar, complex mathematics they are learning in CE 506 to concepts they have already become accustomed to in the past. Due to its length, the derivation of Bernoulli's equations is only included in section A.3 of the appendix; however, the question from the assignment follows below.

HW 5, problem 3

Consider an Eulerian description of the flow of a fluid. The flow is characterized by the triple (v, ρ, σ). The flow is said to be *potential* if the velocity is derivable as the gradient of a scalar field ψ such that $v = grad(\psi)$. The body force field acting on the fluid is said to be be conservative if there is also a potential U such that $f = -\rho grad(U)$. The special case in which the stress σ is of the form $\sigma = -pI$ where p is a scalar field, is called the *pressure* field (p is the fluid pressure or hydrostatic pressure).

1. Show that for potential flow, a pressure field $\sigma = -pI$, and conservative body forces, the momentum equations imply that:

$$\frac{\partial \psi}{\partial t} + \frac{1}{2} v \cdot v + U + \frac{1}{\rho} p = \tau(t)$$

   This is Bernoulli's equation for potential flow.

2. If the motion is steady and $U = gz$ where g is the acceleration due to gravity and z is the elevation above a reference plane show that the equations in (1) reduce to:

$$\frac{1}{2} \rho v \cdot v + \rho g h = \tau$$

   where $h = z + \frac{p}{\rho g}$ is the hydraulic head (the sum of the elevation z and the pressure head).

3.2 Coding Project

The addition of a new coding project represents the most significant upgrade to the course since Dr. Scott first began teaching the class. The new project requires students to implement their own finite element code that will solve two-dimensional linear elastostatics problems. In completing this project, the students have moved on from the one-dimensional code that solved somewhat trivial problems to a code that solves problems which, though quite simple, are not trivial. In addition, the two dimensional nature of the problem lends itself to a more meaningful visualization of solutions. This has the potential to let the students see how they may use their code to increase their understanding of the situation they are modeling. Out of the textbook for the class the Galerkin formulation of the problem is as follows [5]:

Given $f, g,$ and h, find $u^h = v^h + g^h \in S^h$ such that for all $w^h \in V^h$

$$a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h)$$

The Galerkin formulation shown above represents a generalized framework for problems of this type. In this class Dr. Scott has given a more concrete problem description for what the students are to solve with their finite element code. The description for the first problem of the project is given in Figure 3.1.

In Figure 3.2 we see a mesh that represents the set up of the problem before any calculations are performed. In this problem rather than just looking at a line of nodes, as would be the case in a one-dimensional problem, here we see nodes arranged nicely in a square. It is easy for students to imagine this representing some square piece of material. Figure 3.3 displays the mesh after the student has carried out a displacement of 1.0 on the right boundary of the object with values set for the material modulus of elasticity and Poisson's ratio. Again, it is easy for students to imagine that if such a displacement were actually carried out on a physical model of the material, it would take the shape seen in this figure. Once students have successfully completed this phase of the project, they are free experiment with other values for material parameters, loadings, and displacements. This will likely give the students an opportunity to experience the power that computer simulation offers to discover new concepts about the specific problem they are simulating. In this project, a student can effectively confirm the theoretical limitations placed on Poisson's ratio, for instance. The ability of the students to better visualize what types of problems and behavior their code is actually simulating represents a significant upgrade to the learning of the students and will likely lead to an increase in their interest in pursuing research in this area.

3.3 Problem-Oriented Teaching

In order to overcome the significant increase in the difficulty of the new project and the increased time required for completion, the methods used to teach concepts required to successfully implement working code were taught in a more concrete and problem-oriented manner than in previous semesters. In semesters past the coding concepts were taught right out of the textbook and emphasized the formulas and equations used to derive the finite element method. With the current two-dimensional problem, however, these ideas were taught emphasizing implementation over math.

Consider two examples of notes used for class lectures in teaching the concepts for calculating elemental stiffness matrices. In Figure 3.4 we see lecture notes used by Dr. Scott to teach the concepts required for calculating elemental stiffness matrix for a one-dimensional linear element. In this approach Dr. Scott has started with the integral form of the element stiffness matrix

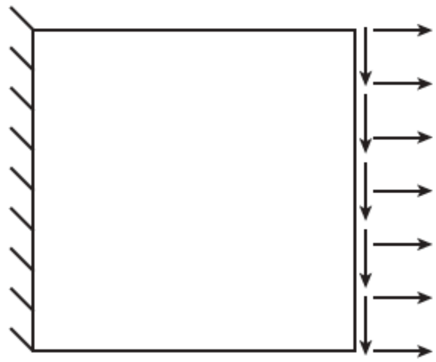

Figure 1: Problem setup for patch tests.

FEA of Linear Elastic Problems

In this assignment you will build a finite element code to solve two-dimensional linear elasticity problems. You will then solve two problems with your code.

Problem 1

The problem definition is shown in Figure 1. In this case, the elastic block is square with $M, N = 1$. The left side of the block is restrained from moving in the x and y directions and the right side is subject to prescribed displacements and tractions in the x and y directions as specified below. The material parameters E and ν will also be specified below. The top and bottom of the block are traction free and there is no body force.

1. In this problem we will solve a simple patch test. Patch tests are commonly used to assess the correctness of a finite element code. We set our displacement boundary conditions on the right end to be $u_x = 1$. We take $E = 1$ and $\nu = 0$. Solve the problem for $p, q = 1, 2$ and $m, n = 1, 4, 32$. Turn in contour plots of each component of the computed displacement field u_x and u_y and each component of the stress σ_{11}, σ_{22}, and σ_{12}.

2. Solve the same problem but now set a traction boundary condition $t_x = 1000$, $t_y = 0$, $E = 1e7$, and $\nu = 0.3$. The material parameters are an accurate model for steel. Solve the problem for $p, q = 1, 2$ and generate a mesh of sufficient resolution to resolve the physics of the problem. Turn in contour plots of each component of the computed displacement field u_x and u_y and each component of the stress σ_{11}, σ_{22}, and σ_{12} for your converged solution.

Figure 3.1: Problem 1 of the two dimensional coding project

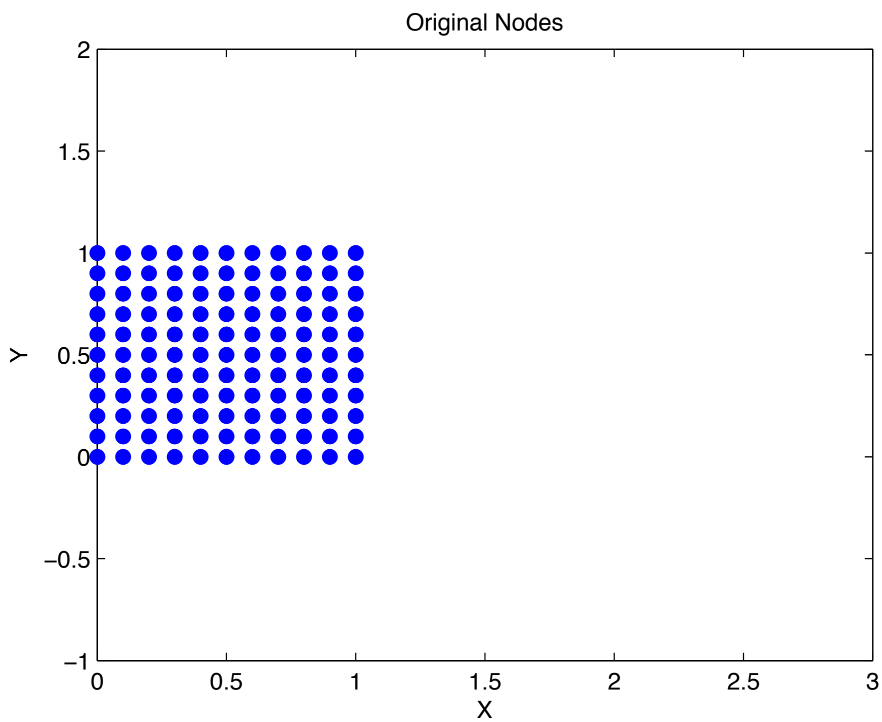

Figure 3.2: Original undeformed mesh, dots represent mesh nodes

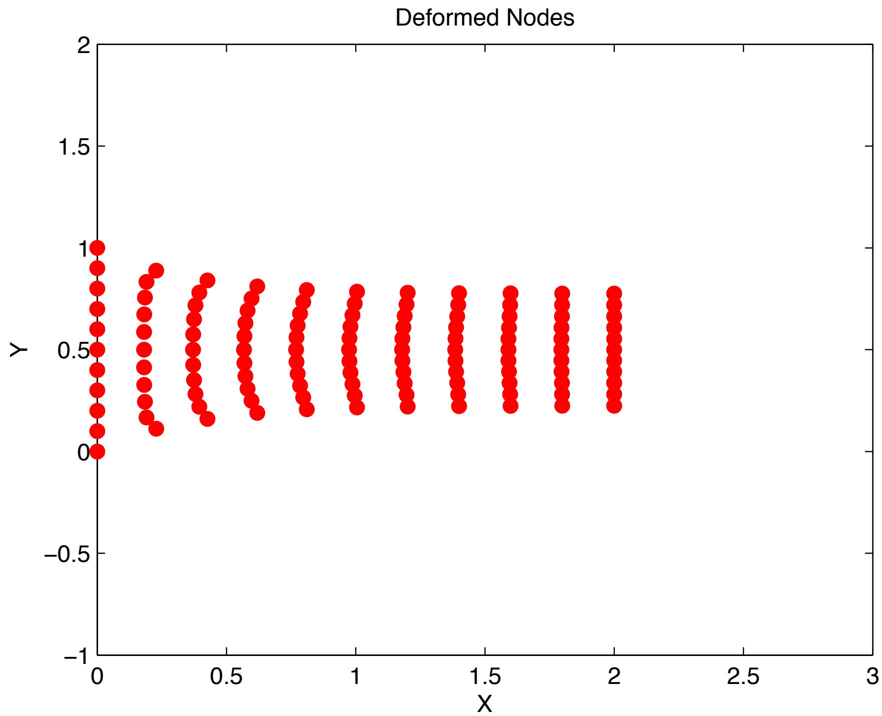

Figure 3.3: Mesh after a boundary displacement of 1, dots represent mesh nodes

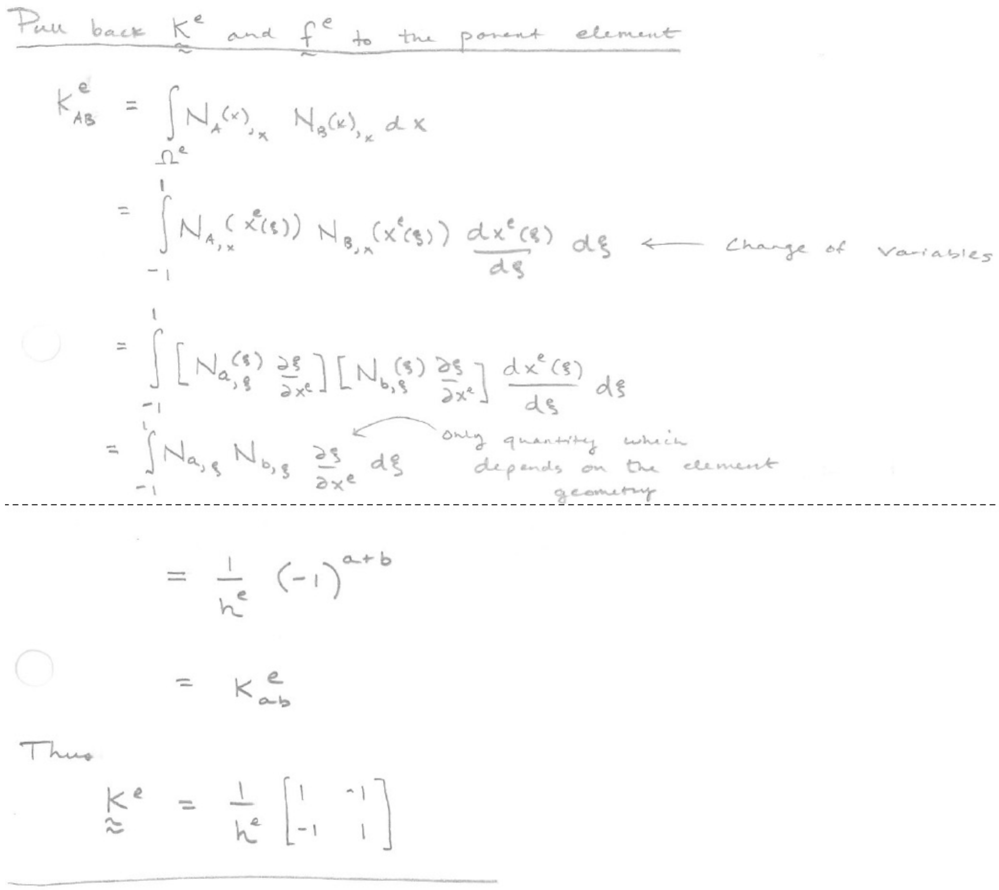

Figure 3.4: one-dimensional teaching of element K

and shown how this integral may be explicitly calculated to give the stiffness matrix. While this is a good example of the methods by which explicit computations can be performed by hand for simple finite element problems, it still will require a significant amount of thought for the students to figure out how to transfer what is shown in Figure 3.4 to a code that will perform the same integration. In Figure 3.5 we see a much different approach. Rather than working with the formulas used in the derivations of the finite element method, the teaching of the calculation of an elemental stiffness matrix is done with a fully written, working algorithm. This approach allows the students to concentrate their efforts immediately on implementing the code correctly rather than losing time transferring mathematical derivations to working code.

```
Algorithm  assembleElementK

Input: An element e
Output: K^e

    D = buildD(E, ν)
  {Pts, Wts} = quadratureRule(n^ξ_int, n^η_int)
    for  igpt = 0, ..., n_int - 1
    {      W = Wts[igpt]
             {N, ∂N/∂ξ, ∂N/∂η} = lagrange2D(ξ_igpt, η_igpt, P, q)
           {N, ∂N/∂x, ∂N/∂y, detJ, x} = lagrange2DSpatial(ξ_igpt, η_igpt, P, q, N
                                                             ∂N/∂ξ, ∂N/∂η, x^e)

           for a = 0, ..., n_en
           {
                 B_a = getB(a, ∂N/∂x, ∂N/∂y)
                 temp = B_a^T * D

                 for b = 0, ..., n_en
                 {
                     B_b = getB(b, ∂N/∂x, ∂N/∂y)
                     N = temp * B_b
                     for i = 0, ..., n_dof - 1
                     {
                          for j = 0, ..., n_dof - 1
                          {
                               r = a * n_dof + i
                               s = b * n_dof + j
                               K^e[r,s] += N[i,j] * W * detJ
                          }
                     }
                 }
           }
    }
```

Figure 3.5: two-dimensional teaching of element K

3.4 Observations so far on upgrades

With each of the upgrades implemented this semester, there are a number of observed challenges and benefits that have come along with them. In considering these challenges and benefits, it should be remembered that all upgrades reported in this project have been implemented "on the fly" while the class was already in progress. Therefore, much of the increased difficulty associated with the introduction of new assignments and projects has likely been compounded by the fact that they were not planned before the semester began. In the same manner, many of the benefits observed this semester may have not been fully realized. In the future, when the course is already designed to include many of these upgrades, the difficulties will likely be less and the benefits may be more fully capitalized on.

In the case of the homework assignments, it seems that the concepts required to successfully complete them were not given enough attention during this teaching of CE 506. This is likely due to the fact the problems were written for Dr. Scott's CE 608 class, in which students received a much more thorough introduction to the topics covered in these homework assignments. In my interactions with the students as they attempted to solve the problems, it was clear to me that though the students did technically have written in their notes the concepts required to solve the assigned problems, these concepts had not been given enough attention to allow the students to become comfortable with them. Because the original purpose of the new homework assignments was to transfer some of the material from CE 608 to CE 506 in preparation for future teachings of CE 608, careful attention ought to be paid to transfer the teaching as well as the assignments.

In my involvement with the past three teachings of this class, it has seemed that though the complexity of the coding assignments has been increasing, the concepts the students struggle with remain the same. For example, as a teaching assistant for the class I most often get asked questions such as, "How big is the force vector?" or "How come my values for the force vector are wrong?" I get asked these questions whether the problem is one-dimensional linear or two-dimensional quadratic. To me, this is strong evidence that a two-dimensional coding project is well within students' reach. Furthermore, once solutions are obtained and plotted by the students, the visualization tends to lead to increased understanding of the problems being solved. Thus, the implementation of a two-dimensional coding project seems to have been successful and I would recommend keeping it as part of the class.

One concern that still remains in regards to the two-dimensional coding project is the time required by the students to successfully complete the project. During this semester's teaching the due date was reset numerous times because the students simply needed more time. Therefore, in the future, the problem-oriented approach to teaching this section of the class needs to continue with a special emphasis on using it to pace the students. This semester's teaching helped boost the students' understanding of the implementation of the code. However, there did not seem to be much effect on the students' motivation to work at a quick pace. In other words, the students seemed to understand well what they needed to do but did not get started on the project right away. In the future if the students are able to see beforehand the time frame for the teaching of each component of the code, as well as appropriately paced due dates for coding benchmarks, they will likely be more successful in completing the code in a timely manner.

CHAPTER 4. CONCLUDING REMARKS

With advances in students' abilities to succeed in CE 506, we will likely see more and more students with the confidence and interest in getting themselves involved with research in computational mechanics. The added student involvement has the potential to create a talented group of professors and students who can collaboratively produce groundbreaking research and establish the BYU civil engineering department as a premier institution for the study of computational mechanics. The progress that Dr. Scott has demonstrated as the teacher of CE 506 and the abilities the students have demonstrated to rise to the challenges presented to them form the beginning of the realization of this goal. This class and the effort that is put into making it a meaningful and worthwhile class to all who participate in it will remain key to producing this type of environment.

This is likely a rare occurrence of this type of project within the civil engineering department; one in which, rather than working on engineering theory or design, the work has been directed toward the teaching and learning of engineering. For me this project represented an unexpected and abrupt change in the path I was taking toward a masters degree. However, ultimately, I feel that I have been better off personally and of more use to the department since I switched to this project. During my time working on this project, Dr. Scott and I have been in close communication. I feel that I have personally benefitted from an increased association with him and the opportunity to look at the effectiveness of a class from the perspective of both teacher and student. Furthermore, there are potentially great benefits to both the department and students with a more frequent use of this type of project. It takes some of the pressure off faculty for the responsibility of producing world-class teaching of its classes and allows the opportunity for a current student to take partial ownership of this task. Also, it gives a meaningful opportunity for any students who, like me, discover late in the game a stronger interest in effective teaching and learning over traditional engineering research while still earning an engineering degree.

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APPENDIX A.

A.1 Responses on Good Teaching

Students were asked to respond to the following prompt:

Think of a teacher or two you've had at some point in your education that stands out from the rest because they were talented at teaching. This can be a teacher from any grade level, college included, and from any subject but I'd like them to be official school teachers.

What effect did their good teaching have on you as a student?

- "I wouldn't say that it necessarily changed how I acted as a student as far as whether I got assignments done sooner or performed better, but it did make me like the subject much more, and therefore enjoy the assignments more and usually feel like I learned a lot from lecture.

  The main thing is that they helped me think the subject was cool or interesting whereas if they weren't a good teacher I would have thought the subject lame or boring. Things like computational theory, or math come to mind.

  Also a good teacher made it so that even if I struggled to get a good grade I still enjoyed learning the concepts and lectures"

  -Computer Science, Graduate Student

- "The best teacher I ever had was my AP US History teacher, Mr. Stockton. No one else in the department wanted the hassle of teaching the class, so he took classes to get certified for AP. He expected more of us than almost any teacher in the school (remember, my high school's graduation rate is under 60%). We actually had to read the textbook and take good notes to do well. He came to class more than prepared to answer questions and get us ready for the test. He started thought-provoking discussions that helped us see the connections between the past, present, and future. It was inspiring. The effort he put into lessons and discussions showed us he cared, and that made us push harder.

Bottom line: Mr. Stockton sacrificed his time to get certified and to prepare for each class. He expected more of us, so we gave him more.

Irrelevant side note: Mr. Stockton also had some of the best classroom management I have ever witnessed. We weren't allowed to get away with anything, but everyone loved him anyway. He joked every week about how we had to come back to Rockford when we were rich and famous and build him a pool for the backyard to thank him for his efforts. At the end of the year, everyone chipped in so some people in the class could fill up a kiddie pool and put it in his yard with a giant card.

-Social Science Teaching

- "I was more excited about learning and participating in class, I felt obligated to do better in their classes, I talked more about what I was learning in class outside of class."

-Medical Student

- "I had a wonderful Portuguese Grammar and Literature teacher in grade 10 that I'll never forget. I liked her as a person, because she had a hilarious personality, but I loved her as a teacher because she was passionate about everything that she taught. She made every lesson seem fascinating, because SHE thought Portuguese grammar was fascinating. She also incorporated her humour in her teaching, which was great because then the lessons became fascinating AND entertaining :) It had an extremely positive effect on me because it taught me to always use your own personality, or the skills you already naturally possess, in your teaching, as well as to teach what you love, and do it with gusto."

-International Relations

- "I've had a few good teachers at BYU. All of them have had a few things in common.

  First all of them were very excited about their respective fields and research. That excitement rubbed off on us in the classroom. Their passion often revealed itself as they would talk about the applications of the knowledge we were acquiring. Sometimes their passion resulted in me liking the subject more, sometimes it was just good enough to keep myself awake and attentive. Poor teachers often fail to grab my attention, which is critical in any teaching/learning situation.

Second, the teachers were all on the same level as the students. They understood our needs, understood what we understood and more importantly what we didn't understand. Their awareness of our needs allowed them to walk step by step along with us in the learning process, and helped us feel comfortable asking questions. Poor teachers I've had have taught either way too simply and slowly (this would be boring) , or way over our heads (this would be discouraging). The real question to evaluate teachers on this point is: Do the students feel comfortable asking questions and occasionally looking stupid? If not, the teacher may be brilliant, but ultimately unable to transfer knowledge."

-Mechanical Engineering

- "Dr. Gilchrist of Political Philosophy. I took his GE required course without knowing if I'd like the material. I found that I really enjoyed learning about it and I think because of his enthusiasm. He made the material seem very relevant and the things he taught still linger in my mind. I was excited for each new topic because he made it seem to me as if we were opening a new treasure chest like in a Zelda game, or a present at Christmas. I'm left with the feeling that I'd like to spend more time reviewing what he taught and related material. I was motivated to email him about some of his spin-off subjects."

-Biomedical Engineering, Graduate Student

- "The teachers that I've appreciated the most were the ones that focused on teaching the students, and not the subject. The subject was a vehicle for teaching us. I appreciate a teacher who is engaging and inspirational. One who motivates an interest in the subject. This is done by interesting examples and experiences which show an application of the subject.

  I appreciate a teacher who asks questions and initiates a class discussion, and who also stimulates an environment that prompts students to ask questions. I also appreciate a teacher who is always seeking to gauge students current understanding so as to know what to emphasize more or less.

The main effect this had on me was inspiring a personal interest in the subject. The result of this personal interest is a permanent knowledge of the subject. I look forward to learning and listening to a good teacher"

-Mechanical Engineering, Graduate Student

- There are a few that stand out to me over both my primary and secondary educational experience. The main thing that I noticed was a desire to learn the material they were teaching. I wanted to go to class because I knew the lectures would be great and I knew I would walk away from that hour feeling like it was actually worth going to class. Then when I got home I would want to study the material and do the homework. So in essence they cultivated in me a desire to learn. It didn't just feel like a check off the list towards a career. I also feel I gained a better understanding of the subject because of this. Those classes are the ones that I can still look back and draw the info back out of the recesses of my mind. Other classes were almost entirely forgotten within a few weeks of the final test.

  -Dental Student

- "I think something interesting about my favorite teachers is that many of them are from my hardest classes and I didn't do especially well in them. One teacher taught really well with the textbook and it was very simple to follow along in the class and that way build on what we learned. They set high expectations but are also aware of limitations and are willing to work with you. Because of that you learn more than just the material but how to problem solve and be diligent. Good teachers inspire me to be better and use the knowledge I gain to help others."

  -Illustration

- "Good teaching not only involves being able to convey the information in a way that students understand it, but also to inspire the students to be better people. I think that being able to put things in their proper perspective in a way that others can see is a very important skill. I am grateful for the teachers who have been able to help me see things in a new way. This often involves bringing in a discussion of gospel.

I've had another professor that had impeccable handwriting on the white board. Everything he put up on the board had meaning and was clear. His lecture followed right along with it.

Any teacher that can take an abstract subject and present it in a way that is easily understandable"

-Mechanical Engineering, Graduate Student

- "I have a lot I could say on this because I think the teacher makes a HUGE difference in what and how you learn, so I'll try to summarize my thoughts.

  The first example that came to mind was my professor for multi-variable calculus. Basically I wasn't looking forward to the class and was kind of nervous because I hadn't taken calculus in years and had just gotten back from my mission and I'd hear this class was pretty hard etc etc, but then I had the best teacher ever and it felt like the easiest math class I'd ever taken! He just taught so clearly and simply and in such an organized fashion. He would teach a specific principle, then work through a problem that applied that principle, then assign a few similar problems for us to work through on our own. It was all just so straight-forward and clear that he made it easy to learn. I think a great teacher is someone who is an expert in his/her field but can explain a concept so simply and clearly that any non-expert can understand it.

  The second example that comes to mind was my fourth grade teacher Mr. Courtney. He was that stereotypical example of a "fun" teacher. We learned a lot in his class but he would spice things up by putting things into songs and tap dancing on the desks, etc. Everybody loved him and everyone loved class because he made his students excited to learn and showed that learning really can be a lot of fun.

  The last examples I thought of would be my current professors. I have 4 and I like 2 and dislike 2. Long story short, the two I like are overall good people and seem to care about their students. One has even memorized all of our names (all 110). You can tell they care about helping their students succeed. Feeling that from them makes me as a student care a lot more about class and what I'm contributing. I feel like I enjoy the classes more, get more out of lectures, try harder on the readings, etc. I think learning from someone is a lot easier and more natural when you feel like that person actually cares about you and is really trying to help you. These two professors also just seem like really good, honest people. They are easy to respect and look-up to. The other two make questionable comments at times, occasionally swear, have very strong viewpoints and make them known while also speaking very condescendingly of differing opinions. As a student, I have a hard time respecting them, I don't trust what they say, and I have a harder time caring about what they're teaching. When you teach, who you are really shines through and I feel like teachers should be role models that we can look-up to and try to emulate.

-Law Student

- "I don't know if I can give you an answer that's not long or comprehensive! I'd have to say Dr. Williams stands out from the rest. He's my committee chair and I worked for him for 1.5 years, so I know him very well. He has a unique teaching style in which he lets us create contracts for what we want to do during that semester. There are certain guidelines/requirements, but he leaves it up to us to pick what specific things we want to read. I like having that type of autonomy. I feel like it's helped me take more responsibility for my learning, in part because I'm motivated and interested in learning the things I pick. I like that we have to create a contract and schedule for when we will turn things in. There are no set due dates, except that everything has to be turned in by the end of finals week. He puts a lot of trust in his students to make those decisions, and I love that. He's not into busy work or micromanaging what we do. He rarely lectures. Instead, he has the students in the class teach each other from what they've read. He facilitates class discussions by asking questions. He also has us work on real world projects and report back on those things in class. That is also motivating.

  I know several people struggle with that approach at first because they don't know what is expected of them. But once they realize he just wants you to learn and become something from the class, they like it more. I like that he spends a lot of time on spiritual thoughts and that he encourages us to evaluate ourselves as learners. Our final includes several short papers at the end of the semester which encourage us to reflect on what we learned in the class and how those things have helped us to become more than we were. I love that! I like the reflection he encourages."

-Instructional Psychology and Technology, Graduate Student

- "The teacher asked good questions that opened the door to deeper understanding. Rather than just giving spoon-fed information this teacher instilled curiosity and helped guide us to think how to apply the information being taught."

-Civil Engineering, Graduate Student

- I remember several teachers that really made a difference for me. They were respectful of their students and cared first and foremost about their learning. They were fair in their grading and generous in their feedback. They were there for me when I needed them. That is the biggest thing: they were there.

-English Language

- The first was Mr. Gehant, my Geometry teacher in 9th grade. Mr. Gehant was enthusiastic and inventive when it came to teaching math. He generally divided us in groups and had us work together on problems. The day that stands out most when we came in one day and he refused to speak anything but french. He taught an entire lesson on ratio's in french using pie. At the end he wrapped everything up in english. Overall it was his excitement and enthusiasm that made each day interesting. I actually had a lot of good teachers that year, and they made me want to be a high school teacher. I didn't entirely stick to that, but their love of teaching was certainly contagious.

  The second was Dr. Kauwe, he was my introductory Biology professor my sophomore year of college. He also had a passion for science and an energy in his teaching. Honestly one of the things I appreciated was his candor with scientific topics, when we occasionally discussed things like evolution or global warming where some students raised doubts, he was unapologetic and clear on what the science is. He also was an effective teacher, using activities and group work to apply concepts. I decided to switch my major to Biology in part because of how interesting he made science (I hadn't had a good science teacher since 7th grade), and when I went to chat with him about the idea he was available and gave me lots of good advice.

So that's how I decided to become a professor. I've had lots of other good teachers in my life, but I think all of them where excited about what they were teaching and interacted with students without condescension or trepidation.

-Biology, Graduate Student

- There is one teacher who stands out to me as having an effect on me when I was a student: my High School Biology teacher, Mr. Bowen. I think the reason why he had an effect on me was because he was deeply passionate about the subject he was teaching and his passion rubbed off on me.

  Mr. Bowen would come into class every day totally psyched to be teaching his students about DNA, genes, or cellular biology, or whatever. He also had hobbies like collecting mushrooms in the hills around San Jose and selling them to restaurants. I did not get a great grade in the class because I was not a very good student when I was in High School, and actually had to end up taking it again my Sophomore or Junior year I think, where I did a little better the second time. But even though my grade the first time around was not very good, it was the best class I took in High School and I can trace a lot of my general understanding and interest in science back to his class.

  As a sidenote, when I was a teacher in Japan, I learned very fast that I was most effective as a teacher when I was having fun teaching. If I was having fun and was excited about what I was teaching, the students were engaged. If I wasn't, then I would lose their interest.

-Law Student

A.2 Homework Assignment 4 - vectors, index notation, bases, tensor calculus [1]

1. Given the vectors:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \mathbf{c} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

calculate using index notation:

   (a) $\mathbf{a} \cdot \mathbf{b}$

   (b) $\mathbf{a} \times \mathbf{b}$

   (c) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$

2. Given a plane $\mathbf{P}$ with normal $\mathbf{n} = 1\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}$ and the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, calculate the projection of $\mathbf{v}$ onto $\mathbf{P}$.

3. Calculate $\delta_{ij}\delta_{ij}$ using rules of index notation and the definition of the Kronecker delta.

4. (a) Given a vector $\mathbf{x}$ with representation in the standard basis

$$\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

find its representation in a new basis, $\mathbf{V}$, with basis vectors:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$$

(b) Repeat part (a) with the standard representation for $\mathbf{x}$ given by $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and with the basis $\mathbf{V}$ given by basis vectors:

$$\mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5. Consider the scalar field $\varphi(\mathbf{x}) = (x_1)^2 x_3 + x_2 (x_3)^2$ and the vector field $\mathbf{v(x)} = x_3 \mathbf{e}_1 + x_2 \sin(x_1) \mathbf{e}_3$. Find the components of $\bigtriangledown \varphi(\mathbf{x})$ and $\bigtriangledown \mathbf{v(x)}$

6. Use index notation to prove the following where $\mathbf{A}$ is a constant second-order tensor:

   (a) $\bigtriangledown \mathbf{x} = \mathbf{I}$

   (b) $\bigtriangledown \cdot \mathbf{x} = 3$

   (c) $\bigtriangledown (\mathbf{x} \cdot \mathbf{A}\mathbf{x}) = (\mathbf{A} + \mathbf{A}^T)\mathbf{x}$

A.2.1 Solution to HW 4

1.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \mathbf{c} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

(a)

$$\mathbf{a}\cdot\mathbf{b} = (a_i\mathbf{e}_i)\cdot(b_j\mathbf{e}_j)$$

$$= a_ib_j(\mathbf{e}_i\cdot\mathbf{e}_j)$$

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij} \qquad \text{Identity}$$

$$\mathbf{a}\cdot\mathbf{b} = a_ib_j\delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 \text{ when } i = j \\ 0 \text{ when } i \neq j \end{cases} \qquad \text{So we can switch the } j \text{ index to } i$$

$$\mathbf{a}\cdot\mathbf{b} = a_ib_i \qquad \text{The repeated index implies a sum}$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

$$= (1)(1) + (2)(3) + (3)(-2)$$

$$\mathbf{a}\cdot\mathbf{b} = 1$$

(b)

$$\mathbf{a} \times \mathbf{b} = (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j)$$

$$= a_i b_j (\mathbf{e}_i \times \mathbf{e}_j)$$

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k \qquad \text{Identity}$$

$$= a_i b_j \varepsilon_{ijk} \mathbf{e}_k \qquad \text{Sum over the indicies } i, j, k$$

$$= \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{i=1}^{3} a_i b_j \varepsilon_{ijk} \mathbf{e}_k$$

$$\varepsilon_{ijk} = \begin{cases} 0 \text{ if any of } i, j \text{ or } k \text{ are the same} \\ +1 \text{ if } ijk \text{ is 123, 231, or 312} \\ -1 \text{ if } ijk \text{ is 132, 321, or 213} \end{cases}$$

$$\mathbf{a} \times \mathbf{b} = a_3 b_2 \varepsilon_{321} \mathbf{e}_1 + a_2 b_3 \varepsilon_{231} \mathbf{e}_1 + a_3 b_1 \varepsilon_{312} \mathbf{e}_2 ...$$

$$+ a_1 b_3 \varepsilon_{132} \mathbf{e}_2 + a_2 b_1 \varepsilon_{213} \mathbf{e}_3 + a_1 b_2 \varepsilon_{123} \mathbf{e}_3$$

$$= (3)(3)(-1)\mathbf{e}_1 + (2)(-2)(1)\mathbf{e}_1 + (3)(1)(1)\mathbf{e}_2 ...$$

$$(1)(-2)(-1)\mathbf{e}_2 + (2)(1)(-1)\mathbf{e}_3 + (1)(3)(1)\mathbf{e}_3$$

$$= (-9-4)\mathbf{e}_1 + (3+2)\mathbf{e}_2 + (-2+3)\mathbf{e}_3$$

$$\mathbf{a} \times \mathbf{b} = -13\mathbf{e}_1 + 5\mathbf{e}_2 + \mathbf{e}_3 = (-13, 5, 1)$$

(c)

$$
\begin{aligned}
\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= a_i\mathbf{e}_i\cdot(b_j\mathbf{e}_j\times c_k\mathbf{e}_k) \\
&= a_i\mathbf{e}_i\cdot[b_jc_k(\mathbf{e}_j\times\mathbf{e}_k)] \\
&= a_i\mathbf{e}_i\cdot[b_jc_k\varepsilon_{jkl}\mathbf{e}_l] && \text{Identity} \\
&= a_ib_jc_k\varepsilon_{jkl}(\mathbf{e}_i\cdot\mathbf{e}_l) \\
&= a_ib_jc_k\varepsilon_{jkl}\delta_{il} && \text{Identity} \\
&= b_jc_ka_l\varepsilon_{jkl} && \text{Switch index } i \text{ to } l \\
&= b_3c_2a_1\varepsilon_{321}+b_2c_3a_1\varepsilon_{231}+b_3c_1a_2\varepsilon_{312}... \\
&\quad +b_1c_3a_2\varepsilon_{132}+b_1c_2a_3\varepsilon_{123}+b_2c_1a_3\varepsilon_{213} \\
&= (-2)(-1)(1)(-1)+(3)(0)(1)(1)+(-2)(-2)(2)(1)... \\
&\quad +(1)(0)(2)(-1)+(1)(-1)(3)(1)+(3)(-2)(3)(-1)
\end{aligned}
$$

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = 21$$

2.

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \qquad \text{normalize } \mathbf{n}$$

Let $\mathbf{v}_P$ be the projection of $\mathbf{v}$ onto $\mathbf{P}$

$$\begin{aligned}
\mathbf{v}_P &= \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) \\
&= \mathbf{n} \times \left(0, \frac{-5}{\sqrt{6}}, \frac{10}{\sqrt{6}}\right) \\
&= \left(\frac{25}{6}, \frac{5}{3}, \frac{-5}{6}\right)
\end{aligned}$$

OR

$$\begin{aligned}
\mathbf{v}_P &= \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \\
&= \mathbf{v} - \left(\frac{-7}{\sqrt{6}}\right)\mathbf{n} \\
&= \mathbf{v} - \left(\frac{-7}{6}, \frac{7}{3}, \frac{-7}{6}\right) \\
&= \left(\frac{25}{6}, \frac{5}{3}, \frac{-5}{6}\right)
\end{aligned}$$

3.

$$\begin{aligned}
\delta_{ij}\delta_{ij} &= (\mathbf{e}_i \cdot \mathbf{e}_j)(\mathbf{e}_i \cdot \mathbf{e}_j) \\
&= \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

4. (a) The basis vectors may be assembled as columns of a basis change matrix, $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

Let $\mathbf{x}_V$ be the representation of the vector in the basis $\mathbf{V}$. Then a matrix equation can be set up relating the two vectors and the new representation found:

$$
\begin{aligned}
\mathbf{x} &= \mathbf{V}\mathbf{x}_V \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix} \\
\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix} \\
\begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix} \\
\begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix} &= \begin{bmatrix} av_{11} + bv_{21} + cv_{31} \\ av_{12} + bv_{22} + cv_{32} \\ av_{13} + bv_{23} + cv_{33} \end{bmatrix}
\end{aligned}
$$

(b) We now have a basis change matrix given by:

$$
\mathbf{V} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

The matrix equation is now:

$$
\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix}
$$

$$
\begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} x_{V1} \\ x_{V2} \\ x_{V3} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}
$$

5. The gradient of the scalar field:

$$
\begin{aligned}
\varphi(\mathbf{x}) &= (x_1)^2 x_3 + x_2 (x_3)^2 \\
\triangledown \varphi(\mathbf{x}) &= \frac{\partial \varphi}{\partial x_1}\mathbf{e}_1 + \frac{\partial \varphi}{\partial x_2}\mathbf{e}_2 + \frac{\partial \varphi}{\partial x_3}\mathbf{e}_3 \\
&= (2x_1x_3)\mathbf{e}_1 + (x_3^2)\mathbf{e}_2 + (x_1^2 + 2x_2x_3)\mathbf{e}_3 \\
&= (2x_1x_3, x_3^2, x_1^2 + 2x_2x_3)
\end{aligned}
$$

The gradient of the vector field:

$$
\begin{aligned}
\mathbf{v}(\mathbf{x}) &= x_3\mathbf{e}_1 + x_2\sin(x_1)\mathbf{e}_3 \\
\bigtriangledown(\mathbf{x}) &= \frac{\partial v_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j \\
&= \frac{\partial v_1}{\partial x_1}\mathbf{e}_1\otimes\mathbf{e}_1 + \frac{\partial v_2}{\partial x_1}\mathbf{e}_2\otimes\mathbf{e}_1 + \frac{\partial v_3}{\partial x_1}\mathbf{e}_3\otimes\mathbf{e}_1 + \frac{\partial v_1}{\partial x_2}\mathbf{e}_1\otimes\mathbf{e}_2 + \frac{\partial v_2}{\partial x_2}\mathbf{e}_2\otimes\mathbf{e}_2 + \dots \\
&\quad \dots\frac{\partial v_3}{\partial x_2}\mathbf{e}_3\otimes\mathbf{e}_2 + \frac{\partial v_1}{\partial x_3}\mathbf{e}_1\otimes\mathbf{e}_3 + \frac{\partial v_2}{\partial x_3}\mathbf{e}_2\otimes\mathbf{e}_3 + \frac{\partial v_3}{\partial x_3}\mathbf{e}_3\otimes\mathbf{e}_3 \\
&= 0+0+x_2\cos x_1\mathbf{e}_3\otimes\mathbf{e}_1 + 0+0+\sin(x_1)\mathbf{e}_3\otimes\mathbf{e}_2 + (1)\mathbf{e}_1\otimes\mathbf{e}_3 + 0+0 \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ x_2\cos(x_1) & \sin x_1 & 0 \end{bmatrix}
\end{aligned}
$$

6. $\mathbf{x} = (x_1, x_2, x_3)$

   (a) Prove $\bigtriangledown\mathbf{x} = \mathbf{I}$

$$
\begin{aligned}
\bigtriangledown\mathbf{x} &= \bigtriangledown(x_1, x_2, x_3) \\
&= \frac{\partial x_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j \\
&= \frac{\partial x_1}{\partial x_1}\mathbf{e}_1\otimes\mathbf{e}_1 + 0+0+0+\frac{\partial x_2}{\partial x_2}\mathbf{e}_2\otimes\mathbf{e}_2 + 0+0+0+\frac{\partial x_3}{\partial x_3}\mathbf{e}_3\otimes\mathbf{e}_3 \\
&= \delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \\
&= (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}
$$

(b) Prove $\bigtriangledown \cdot \mathbf{x} = 3$

$$\begin{aligned}
\bigtriangledown \cdot \mathbf{x} &= \bigtriangledown \cdot (x_1, x_2, x_3) \\
&= \frac{\partial x_i}{\partial x_i} = x_{i,i} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

(c) Prove $\bigtriangledown(\mathbf{x} \cdot \mathbf{A}\mathbf{x}) = (\mathbf{A} + \mathbf{A}^T)\mathbf{x}$

$$\begin{aligned}
\bigtriangledown(\mathbf{x} \cdot \mathbf{A}\mathbf{x}) &= \bigtriangledown[x_i \mathbf{e}_i \cdot A_{jk} \mathbf{e}_j \otimes \mathbf{e}_k (x_l \mathbf{e}_l)] \\
&= \bigtriangledown(x_i A_{jk} x_l \delta_{kl} \delta_{ij}) && \text{Change } k \text{ to } l \text{ and } i \text{ to } j \\
&= \frac{\partial}{\partial x_m}(x_j A_{jl} x_l) \\
&= \left[\frac{\partial x_j}{\partial x_m} A_{jl} x_l + \frac{\partial x_l}{\partial x_m} A_{jl} x_j\right] \mathbf{e}_m \\
&= [\delta_{jm} A_{jl} x_l + \delta_{lm} A_{jl} x_j] \mathbf{e}_m && \text{Change the left } j \text{ to } m \text{ and the right } l \text{ to } m \\
&= A_{ml} x_l \mathbf{e}_m + A_{jm} x_j \mathbf{e}_m \\
&= [\mathbf{A} + \mathbf{A}^T]\mathbf{x}
\end{aligned}$$

A.3 Homework Assignment 5: Cauchy stress, balance of linear momentum

Assignment 5

1. The Cauchy stress tensor in a body B is $\boldsymbol{\sigma}(\mathbf{x}, t) = \sigma_{ij}(\mathbf{x}, t)\mathbf{e}_i \otimes \mathbf{e}_j$ where

$$\sigma_{ij}(\mathbf{x}, t) = e^{10-10t} \begin{bmatrix} 10000x_1^2 - 7000x_1x_2 & 7000x_1x_3 & 2000x_3^2 \\ 7000x_1x_3 & 3000x_2^2 & 100x_1 \\ 2000x_3^2 & 100x_1 & 1000x_3^2 \end{bmatrix}$$

   (a) At point $\mathbf{x} = (1, 1, 1)$ at time $t = 1$, compute the stress vector $\mathbf{t}(\mathbf{n})$ in the direction $\mathbf{n} = n_i\mathbf{e}_i$, $n_i = \frac{1}{\sqrt{3}}$, $i = 1, 2, 3$.

   (b) At $t = 1$, what is the total contact force on the plane surface $x_1 = 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1$?

2. Let $\boldsymbol{\sigma}(\mathbf{x}, t)$ be the Cauchy stress. If $\mathbf{n}$ is a direction (a unit vector) such that $\boldsymbol{\sigma}\mathbf{n} = \hat{\sigma}\mathbf{n}$ then $\hat{\sigma}$ is called a principal stress and $\mathbf{n}$ is a principal direction of $\boldsymbol{\sigma}$ (i.e., eigenvalues and eigenvectors of $\boldsymbol{\sigma}$). Let Γ be a plane through a point $\mathbf{x}$ with unit normal $\mathbf{n}$. The *normal stress* $\boldsymbol{\sigma}_n$ at $\mathbf{x}$ is $\boldsymbol{\sigma}_n = (\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n})\mathbf{n}$ and the *shear stress* is $\boldsymbol{\sigma}_t = \boldsymbol{\sigma}\mathbf{n} - \boldsymbol{\sigma}_n$. Show that if $\mathbf{n}$ is a principal direction of $\boldsymbol{\sigma}$, then $\boldsymbol{\sigma}_t = 0$.

3. Consider an Eulerian description of the flow of a fluid. The flow is characterized by the triple $(\mathbf{v}, \rho, \boldsymbol{\sigma})$. The flow is said to be *potential* if the velocity is derivable as the gradient of a scalar field ψ such that $\mathbf{v} = \operatorname{grad}\psi$. The body force field acting on the fluid is said to be *conservative* if there is also a potential U such that $\mathbf{f} = -\rho \operatorname{grad} U$. The special case in which the stress $\boldsymbol{\sigma}$ is of the form $\boldsymbol{\sigma} = -p\mathrm{I}$ where p is a scalar field, is called the *pressure* field (p is the fluid pressure or hydrostatic pressure).

   (a) Show that for potential flow, a pressure field $\boldsymbol{\sigma} = -p\mathbf{I}$, and conservative body forces, the momentum equations imply that

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + U + \frac{1}{\rho}p = \tau(t).$$

   This is Bernoulli's equation for potential flow.

   (b) If the motion is steady and $U = gz$ where g is the acceleration due to gravity and z is the elevation above a reference plane show that the equations in (a) reduce to

$$\frac{1}{2}\rho\mathbf{v} \cdot \mathbf{v} + \rho g h = \tau.$$

   where $h = z + \frac{p}{\rho g}$ is the hydraulic head (the sum of the elevation z and the pressure head).

A.3.1 Solution to HW 5

1. (a) Compute the entries of the Cauchy stress tensor σ with $\mathbf{x}$ = (1,1,1), t = 1

$$\sigma((1,1,1),1) = e^{(10-10(1))}\begin{bmatrix} 10000(1)^2 - 7000(1)(1) & 7000(1)(1) & 2000(1)^2 \\ 7000(1)(1) & 3000(1)^2 & 100(1) \\ 2000(1)^2 & 100(1) & 1000(1)^2 \end{bmatrix}$$
$$= \begin{bmatrix} 3000 & 7000 & 2000 \\ 7000 & 3000 & 100 \\ 2000 & 100 & 1000 \end{bmatrix}$$

Calculate the stress vector via $t(n) = \sigma n$

$$n = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$
$$t(n) = \begin{bmatrix} 3000 & 7000 & 2000 \\ 7000 & 3000 & 100 \\ 2000 & 100 & 1000 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$
$$t(n) = \begin{bmatrix} 6928.2 \\ 5831.2 \\ 1789.8 \end{bmatrix}$$

(b) Start with the total force on a body:

$$F(B,t) = \int_{B_t} f(x,t)dv + \int_{\partial B_t} \sigma(x,n,t)da$$
$$F(B,t) = 0 + \int_{\partial B_t} \sigma n da \qquad \text{No external body forces specified}$$

For this problem think of a cube with center at the origin and limits:

$$-1 \leq x_1 \leq 1$$
$$-1 \leq x_2 \leq 1$$
$$-1 \leq x_3 \leq 1$$

We will integrate over a section of the surface at the value $x_1 = 1$. The vector $n = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a normal vector for the $x_1 = 1$ surface. So by evaluating σn we can get a vector field describing the stress across the entire surface at $x_1 = 1$.

$$\begin{aligned} F(B,t) &= \int_{\partial B_t} \sigma n da \\ &= \int_{\partial B_t} \begin{bmatrix} 10000 - 7000x_2 & 7000x_3 & 2000x_3^2 \\ 7000x_3 & 3000x_2^2 & 100 \\ 2000x_3^2 & 100 & 1000x_3^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} da \text{ with t = 1 and } x_1 = 1 \\ &= \int_{\partial B_t} \begin{bmatrix} 10000 - 7000x_2 \\ 7000x_3 \\ 2000x_3^2 \end{bmatrix} da \end{aligned}$$

We can now change the integral over the boundary surface to a double integral over the limits for the x_2, x_3 directions:

$$F(B,1) = \int_0^1 \int_0^1 \begin{bmatrix} 10000 - 7000x_2 \\ 7000x_3 \\ 2000x_3^2 \end{bmatrix} dx_2 dx_3$$
$$= \int_0^1 \begin{bmatrix} 6500 \\ 7000x_3 \\ 2000x_3^2 \end{bmatrix} dx_3$$
$$= \begin{bmatrix} 6500 \\ 3500 \\ 666.667 \end{bmatrix}$$

This gives a vector representing the total contact force on the surface in each of the x_1, x_2, x_3 directions

2. Since n is a principal direction and unit vector we know that $\sigma n = \hat{\sigma} n$ and $n \cdot n = 1$ so:

$$\begin{aligned} \sigma_n &= (n \cdot \sigma n)n \\ &= (n \cdot \hat{\sigma} n)n \\ &= \hat{\sigma}(n \cdot n)n \\ &= \hat{\sigma} n \\ \sigma_t &= \sigma n - \sigma_n \\ &= (\hat{\sigma} n) - (\hat{\sigma} n) \\ &= 0 \end{aligned}$$

3. Potential flow

$V = grad(\psi)$

$\bigtriangledown \times (\bigtriangledown \psi) = 0 = w$

In other words, the flow is irrotational. The vector w is called the vorticity.

$f = -\rho grad(U)$

$\sigma = -PI$

(a)

$$\begin{aligned} div(\sigma) &= \sigma_{ij,j} \\ &= (-P\delta_{ij})_{,j} \\ &= -P_{,j}\delta_{ij} + 0 \\ &= -P_{,j} \end{aligned}$$

Balance of linear momentum

$$\rho \frac{dv}{dt} = f + div(\sigma)$$

plugging in

$$\rho \frac{dv}{dt} = -\rho grad(U) + -grad(P)$$

Now focusing in on the $\frac{dv}{dt}$ term:

$$\begin{aligned} \frac{d}{dt} v(x(t),t) &= \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial t} + \frac{\partial v}{\partial t} \\ &= \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial t} + \frac{\partial}{\partial t}\frac{\partial \psi}{\partial x_i} \end{aligned}$$

Work on the term: $\frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial t}$

Consider the cross product $v \times w$:

$$\begin{aligned}
v \times w &= \varepsilon_{ijk} v_j \varepsilon_{klm} \frac{\partial v_m}{\partial x_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j \frac{\partial v_m}{\partial x_l} \\
&= \partial v_j \frac{\partial v_j}{\partial x_i} - v_j \frac{\partial v_i}{\partial x_j} \\
&= \frac{1}{2}\frac{\partial (v_j)^2}{\partial x_i} - v_j \frac{\partial v_i}{\partial x_j} \\
\frac{1}{2}\frac{\partial (v_j)^2}{\partial x_i} &= v_j \frac{\partial v_i}{\partial x_j} && \text{Irrotational flow,} w = 0 \\
&= \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial t}
\end{aligned}$$

Rewrite balance of linear momentum equation

$$\begin{aligned}
0 &= \frac{1}{2}\frac{\partial (v_j)^2}{\partial x_i} + \frac{\partial}{\partial t}\frac{\partial \psi}{\partial x_i} + U_{,i} + \frac{1}{\rho}P_{,i} \\
0 &= \frac{\partial}{\partial x_i}\left(\frac{1}{2}(v_j)^2 + \frac{\partial \psi}{\partial t} + U + \frac{P}{\rho}\right) \\
\tau(t) &= \frac{1}{2}v_j^2 + \frac{\partial \psi}{\partial t} + U + \frac{P}{\rho}
\end{aligned}$$

(b) Steady flow gives $\frac{\partial \psi}{\partial t} = 0$

$U = gz$

$$\begin{aligned}
\tau &= \frac{1}{2} v \cdot v + gz + \frac{P}{\rho} \\
&= \frac{1}{2}\rho v \cdot v + \rho g z + P \\
\text{Let } h &= z + \frac{P}{\rho g} \\
\tau &= \frac{1}{2}\rho v \cdot v + \rho g h
\end{aligned}$$

A.4 2D FE code benchmarks

1. **LinearNodes(m, M, n, N)**

   LinearNodes(3, 1, 5, $\frac{5}{3}$), returns:

$$\begin{bmatrix} 0 & 0 \\ 0.333 & 0 \\ 0.667 & 0 \\ 1 & 0 \\ 0 & 0.333 \\ 0.333 & 0.333 \\ 0.667 & 0.333 \\ 1 & 0.333 \\ 0 & 0.667 \\ 0.333 & 0.667 \\ 0.667 & 0.667 \\ 1 & 0.667 \\ 0 & 1 \\ 0.333 & 1 \\ 0.667 & 1 \\ 1 & 1 \\ 0 & 1.333 \\ 0.333 & 1.333 \\ 0.667 & 1.333 \\ 1 & 1.333 \\ 0 & 1.667 \\ 0.333 & 1.667 \\ 0.667 & 1.667 \\ 1 & 1.667 \end{bmatrix}$$

LinearNodes($4, 0.5, 2, 0.25$), returns:

$$\begin{bmatrix} 0 & 0 \\ 0.1250 & 0 \\ 0.2500 & 0 \\ 0.3750 & 0 \\ 0.5000 & 0 \\ 0 & 0.1250 \\ 0.1250 & 0.1250 \\ 0.2500 & 0.1250 \\ 0.3750 & 0.1250 \\ 0.5000 & 0.1250 \\ 0 & 0.2500 \\ 0.1250 & 0.2500 \\ 0.2500 & 0.2500 \\ 0.3750 & 0.2500 \\ 0.5000 & 0.2500 \end{bmatrix}$$

2. **LinearIEN(e, m,** n_{en} **)**

   LinearIEN(e, 3, 4), for e = 0...14

$$\begin{bmatrix} 0 & 1 & 2 & 4 & 5 & 6 & 8 & 9 & 10 & 12 & 13 & 14 & 16 & 17 & 18 \\ 1 & 2 & 3 & 5 & 6 & 7 & 9 & 10 & 11 & 13 & 14 & 15 & 17 & 18 & 19 \\ 4 & 5 & 6 & 8 & 9 & 10 & 12 & 13 & 14 & 16 & 17 & 18 & 20 & 21 & 22 \\ 5 & 6 & 7 & 9 & 10 & 11 & 13 & 14 & 15 & 17 & 18 & 19 & 21 & 22 & 23 \end{bmatrix}$$

LinearIEN(e, 4, 4), for e = 0...7

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\ 6 & 7 & 8 & 9 & 11 & 12 & 13 & 14 \end{bmatrix}$$

3. **Lagrange1D(p, ξ)**

p = 1

Lagrange1D(1, -1), returns:

$$N_a(\xi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

Lagrange1D(1, -0.5), returns:

$$N_a(\xi) = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

Lagrange1D(1, 0), returns:

$$N_a(\xi) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

Lagrange1D(1, 0.25), returns:

$$N_a(\xi) = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

p = 2

Lagrange1D(2, -1), returns:

$$N_a(\xi) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -1.5 \\ 2 \\ -0.5 \end{bmatrix}$$

Lagrange1D(2, -0.5), returns:

$$N_a(\xi) = \begin{bmatrix} 0.375 \\ 0.75 \\ -0.125 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Lagrange1D(2, 0), returns:

$$N_a(\xi) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix}$$

Lagrange1D(2, 0.25), returns:

$$N_a(\xi) = \begin{bmatrix} -0.0938 \\ 0.9375 \\ 0.1562 \end{bmatrix} \quad \frac{dN_a}{d\xi} = \begin{bmatrix} -0.25 \\ -0.5 \\ 0.75 \end{bmatrix}$$

4. **Lagrange2D(p, q, ξ, η)**

p = 1, q = 1

Lagrange2D(1, 1, 0, 1), returns:

$$N_a(\xi,\eta) = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix} \quad \frac{\partial N_a}{\partial \xi} = \begin{bmatrix} 0 \\ 0 \\ -0.5 \\ 0.5 \end{bmatrix} \quad \frac{\partial N_a}{\partial \eta} = \begin{bmatrix} -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \end{bmatrix}$$

Lagrange2D(1, 1, 0.5, 0.5), returns:

$$N_a(\xi,\eta) = \begin{bmatrix} 0.0625 \\ 0.1875 \\ 0.1875 \\ 0.5625 \end{bmatrix} \quad \frac{\partial N_a}{\partial \xi} = \begin{bmatrix} -0.125 \\ 0.125 \\ -0.375 \\ 0.375 \end{bmatrix} \quad \frac{\partial N_a}{\partial \eta} = \begin{bmatrix} -0.125 \\ -0.375 \\ 0.125 \\ 0.375 \end{bmatrix}$$

p = 2, q = 1

Lagrange2D(2, 1, 0, 1), returns:

$$N_a(\xi,\eta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \frac{\partial N_a}{\partial \xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{bmatrix} \quad \frac{\partial N_a}{\partial \eta} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

Lagrange2D(2, 1, 0.5, 0.5), returns:

$$N_a(\xi,\eta) = \begin{bmatrix} -0.0312 \\ 0.1875 \\ 0.0938 \\ -0.0938 \\ 0.5625 \\ 0.2812 \end{bmatrix} \quad \frac{\partial N_a}{\partial \xi} = \begin{bmatrix} 0 \\ -0.25 \\ 0.25 \\ 0 \\ -0.75 \\ 0.75 \end{bmatrix} \quad \frac{\partial N_a}{\partial \eta} = \begin{bmatrix} 0.0625 \\ -0.375 \\ -0.1875 \\ -0.0625 \\ 0.375 \\ 0.1875 \end{bmatrix}$$

p = 2, q = 2

Lagrange2D(2, 2, 0, 1)

$$N_a(\xi,\eta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \frac{\partial N_a}{\partial \xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{bmatrix} \quad \frac{\partial N_a}{\partial \eta} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1.5 \\ 0 \end{bmatrix}$$

Lagrange2D(2, 2, 0.5, 0.5)

$$N_a(\xi,\eta) = \begin{bmatrix} 0.0156 \\ -0.0938 \\ -0.0469 \\ -0.0938 \\ 0.5625 \\ 0.2812 \\ -0.0469 \\ 0.2812 \\ 0.1406 \end{bmatrix} \quad \frac{\partial N_a}{\partial \xi} = \begin{bmatrix} 0 \\ 0.125 \\ -0.125 \\ 0 \\ -0.75 \\ 0.75 \\ 0 \\ -0.375 \\ 0.375 \end{bmatrix} \quad \frac{\partial N_a}{\partial \eta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.125 \\ -0.75 \\ -0.375 \\ -0.125 \\ 0.75 \\ 0.375 \end{bmatrix}$$

NOTE: for benchmarks 5 - 7, each part (a), (b), (c), and (d) directly correspond to the next benchmark part (a), (b), (c), and (d). In other words the same mesh and/or element was used in the function call for all of the (a)'s and another mesh and/or element was used for all the (b)'s and so on.

5. **LagrangeNodes(m, M, p, n, N, q)**

   (a) LagrangeNodes(1, 2, 2, 1, 2, 2), returns:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$$

   (b) LagrangeNodes(1, 2.1, 2, 1, 2, 2), returns:

$$\begin{bmatrix} 0 & 0 \\ 1.05 & 0 \\ 2.1 & 0 \\ 0 & 1 \\ 1.05 & 1 \\ 2.1 & 1 \\ 0 & 2 \\ 1.05 & 2 \\ 2.1 & 2 \end{bmatrix}$$

(c) LagrangeNodes(1, 2.2, 2, 1, 2, 2), returns:

$$\begin{bmatrix} 0 & 0 \\ 1.1 & 0 \\ 2.2 & 0 \\ 0 & 1 \\ 1.1 & 1 \\ 2.2 & 1 \\ 0 & 2 \\ 1.1 & 2 \\ 2.2 & 2 \end{bmatrix}$$

(d) LagrangeNodes(3, 1, 2, 2, 1, 2)

$$\begin{bmatrix} 0 & 0 \\ 0.1667 & 0 \\ 0.3333 & 0 \\ 0.5000 & 0 \\ 0.6667 & 0 \\ 0.8333 & 0 \\ 1.0000 & 0 \\ 0 & 0.2500 \\ 0.1667 & 0.2500 \\ 0.3333 & 0.2500 \\ 0.5000 & 0.2500 \\ 0.6667 & 0.2500 \\ 0.8333 & 0.2500 \\ 1.0000 & 0.2500 \\ 0 & 0.5000 \\ 0.1667 & 0.5000 \\ 0.3333 & 0.5000 \\ 0.5000 & 0.5000 \\ 0.6667 & 0.5000 \\ 0.8333 & 0.5000 \\ 1.0000 & 0.5000 \\ 0 & 0.7500 \\ 0.1667 & 0.7500 \\ 0.3333 & 0.7500 \\ 0.5000 & 0.7500 \\ 0.6667 & 0.7500 \\ 0.8333 & 0.7500 \\ 1.0000 & 0.7500 \\ 0 & 1.0000 \\ 0.1667 & 1.0000 \\ 0.3333 & 1.0000 \\ 0.5000 & 1.0000 \\ 0.6667 & 1.0000 \\ 0.8333 & 1.0000 \\ 1.0000 & 1.0000 \end{bmatrix}$$

6. **LagrangeIEN(e, m, p, q,)**

(a) LagrangeIEN(0, 1, 2, 2) returns:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

(b) See item (a)

(c) See item (a)

(d) LagrangeIEN(e, 3, 2, 2) for e = 0...5, returns:

$$\begin{bmatrix} 0 & 2 & 4 & 14 & 16 & 18 \\ 1 & 3 & 5 & 15 & 17 & 19 \\ 2 & 4 & 6 & 16 & 18 & 20 \\ 7 & 9 & 11 & 21 & 23 & 25 \\ 8 & 10 & 12 & 22 & 24 & 26 \\ 9 & 11 & 13 & 23 & 25 & 27 \\ 14 & 16 & 18 & 28 & 30 & 32 \\ 15 & 17 & 19 & 29 & 31 & 33 \\ 16 & 18 & 20 & 30 & 32 & 34 \end{bmatrix}$$

7. **Lagrange2DSpatial(p, q,** x^e, ξ, η **)**

(a) Lagrange2DSpatial(2, 2, x^0, -1, -1), for x^0 see 5(a) and 6(a)

$$det(J) = 1$$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} -1.5 \\ 2 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial N}{\partial y} = \begin{bmatrix} -1.5 \\ 0 \\ 0 \\ 2.0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$x(\xi, \eta) = 0 \qquad y(\xi, \eta) = 0$$

(b) Lagrange2DSpatial(2, 2, x^0, -1, -1), for x^0 see 5(b) and 6(b)

$$det(J) = 1.05$$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} -1.4286 \\ 1.9048 \\ -0.4762 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial N}{\partial y} = \begin{bmatrix} -1.5 \\ 0 \\ 0 \\ 2.0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$x(\xi, \eta) = 0 \qquad y(\xi, \eta) = 0$$

(c) Lagrange2DSpatial(2, 2, x^0, -1, -1), for x^0 see 5(c) and 6(c)

$$det(J) = 1.1$$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} -1.3636 \\ 1.8182 \\ -0.4545 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial N}{\partial y} = \begin{bmatrix} -1.5 \\ 0 \\ 0 \\ 2.0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$x(\xi,\eta) = 0 \qquad y(\xi,\eta) = 0$$

(d) Lagrange2DSpatial(2, 2, x^0, 1, 1), for x^0 see 5(d) and 6(d)

$$det(J) = 0.0417$$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.0 \\ -12.0 \\ 9.0 \end{bmatrix} \qquad \frac{\partial N}{\partial y} = \begin{bmatrix} 0 \\ 0 \\ 2.0 \\ 0 \\ 0 \\ -8.0 \\ 0 \\ 0 \\ 6.0 \end{bmatrix}$$

$$x(\xi,\eta) = 0.3333 \qquad y(\xi,\eta) = 0.5$$

A.5 2D Coding Project

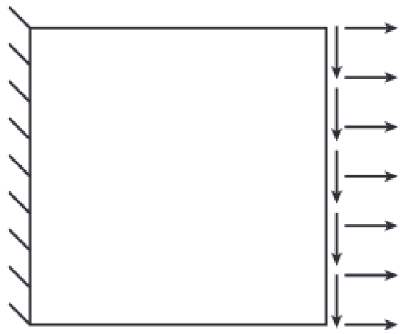

Figure 1: Problem setup for patch tests.

FEA of Linear Elastic Problems

In this assignment you will build a finite element code to solve two-dimensional linear elasticity problems. You will then solve two problems with your code.

Problem 1

The problem definition is shown in Figure 1. In this case, the elastic block is square with $M, N = 1$. The left side of the block is restrained from moving in the x and y directions and the right side is subject to prescribed displacements and tractions in the x and y directions as specified below. The material parameters E and ν will also be specified below. The top and bottom of the block are traction free and there is no body force.

1. In this problem we will solve a simple patch test. Patch tests are commonly used to assess the correctness of a finite element code. We set our displacement boundary conditions on the right end to be $u_x = 1$. We take $E = 1$ and $\nu = 0$. Solve the problem for $p, q = 1, 2$ and $m, n = 1, 4, 32$. Turn in contour plots of each component of the computed displacement field u_x and u_y and each component of the stress σ_{11}, σ_{22}, and σ_{12}.

2. Solve the same problem but now set a traction boundary condition $t_x = 1000$, $t_y = 0$, $E = 1e7$, and $\nu = 0.3$. The material parameters are an accurate model for steel. Solve the problem for $p, q = 1, 2$ and generate a mesh of sufficient resolution to resolve the physics of the problem. Turn in contour plots of each component of the computed displacement field u_x and u_y and each component of the stress σ_{11}, σ_{22}, and σ_{12} for your converged solution.

Problem 2

The problem of a solid circular cylinder subjected to an internal pressure loading is shown in Figure 2. The exact solution, in terms of displacement and stresses in polar

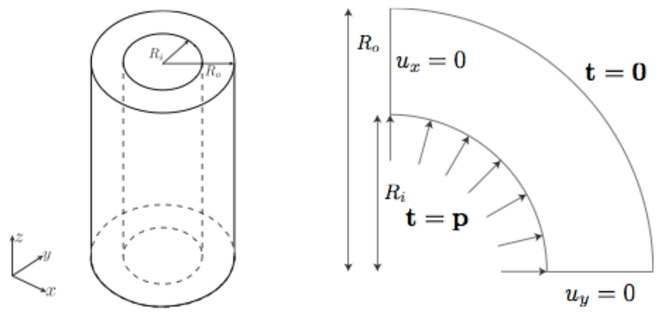

Figure 2: Problem setup for a solid circular cylinder subject to internal pressure loading.

coordinates (R, θ), for the case with constant pressure, is

$$u_R(R) = \frac{1}{E}\frac{PR_i^2}{R_o^2 - R_i^2}\left((1-\nu)R + \frac{R_o^2(1+\nu)}{R}\right) \tag{1}$$

$$\sigma_{RR}(R) = \frac{PR_i^2}{R_o^2 - R_i^2} - \frac{PR_i^2R_o^2}{R^2(R_o^2 - R_i^2)} \tag{2}$$

$$\sigma_{\theta\theta}(R) = \frac{PR_i^2}{R_o^2 - R_i^2} + \frac{PR_i^2R_o^2}{R^2(R_o^2 - R_i^2)}, \tag{3}$$

where R_i is the radius of the inner cylinder, R_o is the radius of the outer cylinder, P is the pressure applied to the inner cylinder, E is Young's modulus, and ν is Poisson's ratio. To solve this problem we take a slice of the cylinder and, employing symmetry, only model one quarter of the resulting annulus as shown in Figure 2 on the right. We will assume the cylinder is infinitely long and employ the plane strain hypothesis which says that

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \tag{4}$$

Confirm that the exact solution is obtained under the limit of mesh refinement (i.e., uniform subdivision of elements) for both polynomial degrees $p, q = 1$ and $p, q = 2$. To do so, it is sufficient to show that the L^2-norm of the error

$$||u - u^h||_{L^2(\Omega)} = \left(\int_\Omega \left(u_R - u_R^h\right)^2 dR\right)^{\frac{1}{2}} \tag{5}$$

goes to zero in the limit of mesh refinement. What is the rate of convergence in terms of the number of degrees of freedom? Generate contour plots of the computed displacements u_R^h and stresses σ_{RR}^h and $\sigma_{\theta\theta}^h$ for $p, q = 1, 2$.