LATERAL TORSIONAL BUCKLING OF LONG, SLENDER, NON-PRISMATIC, SINGLY-SYMMETRIC GIRDERS

by

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GRADUATE COMMITTEE APPROVAL

of a project submitted by

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This project has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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ABSTRACT

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Bridges are expensive to build. Some of the biggest costs involved in bridge construction are labor, material, and crane rental. It is the contractor's challenge to come up with the most economical method of constructing a given bridge in order to have the lowest price to win the bid. Of those costs, the crane size comes into play. The smaller the crane, the less the rental is and the less time it takes to assemble or set up the crane upon arrival at the job site. The intent of this project is to present the findings of a literature search conducted on the lateral torsional buckling capacity of singly symmetric non-prismatic plate girders. The application of such findings is discussed relative to an example of the construction of a particular girder bridge and the crane sizes used.

ACKNOWLEDGMENTS

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1 Introduction

As bridges are built, girders must be lifted into place, usually with a crane. Sometimes the girders are so long that two cranes are required. When the situation necessitates putting up a long slender girder, lateral torsional buckling (LTB) becomes an issue. If the girder is too long or the pick points are too far apart, the girder can not support its own weight while being hoisted into place. To prevent lateral torsional buckling, the unbraced length, the length between the pick points, must be decreased or the top flange must be laterally stiffened. Contractors can place the girder with less cost if smaller cranes can be utilized. However, often the cranes must be placed at opposite ends of the girder making it advantageous to have the pick points closer to the ends of the girder. Engineers must find the balance between crane size and unbraced length. One method of keeping the cranes smaller and the unbraced length long without girder failure is to effectively increase the moment of inertia of the top flange. One such method is securely fastening a horizontal truss on top of the girder between the pick points.

One of the purposes of this project was to use Finite Element Analysis (FEA) to determine the forces needed to be transferred into such a horizontal truss in order to maintain girder stability. Due to many difficulties with the analysis software, which were beyond the scope of this project the final analysis was not completed. A literature search,

the second objective of this project, was conducted and is herein reported. The importance of FEA is discussed with regards to this type of analysis.

1.1 Problem Definition

During construction of a steel girder bridge, the girders to be placed are picked up by a crane. The contractor is responsible for placing the girder without damaging it. For this project, an example will be given of a new girder in the replacement of the Bitchcreek Bridge. This bridge was replaced in spring of 2002 and is located about 15 miles North of Driggs Idaho on the Teton Scenic Byway.

The girders were placed by two cranes simultaneously picking them up from each end. The build-up plate-girders were 197 feet long, 7.5 feet deep and weighed approximately 96,640 pounds each.

With such length involved, lateral torsional buckling became a serious problem. As the crane began to lift one of the girders, it was obvious that the girder was going to fail as the mid span began to bow over, the first sign of lateral buckling. The girder was set back down, and the engineers went to work on another method of lifting the girders. The second time, larger cranes were utilized in order to move the pick points closer together, thus reducing the unbraced length.

This project is geared toward making the first scenario work next time. Initially, one of the intents of this project was to determine the forces necessary to sufficiently stiffen the compression flange of the girder, in the first scenario, to prevent lateral

torsional buckling. This information was then to be used, in part, in the design of a new stiffening truss that could be used on similar future projects.

2 ABAQUS Attempts

In working toward the main objective of this project, a 3D model of the Bitchcreek girder was built in ABAQUS. Material properties were found and assigned. Boundary conditions were given to mimic the lifting forces of the crane. Gravity loads were applied, and many analyses were run. Most times the program failed to run, or ran but reached no convergence. When some analyses were completed, results such as shown in Figure 1 were observed. Figure 2 shows that the maximum differential displacement between the top and bottom amounted to approximately 1/100,000 of a foot.

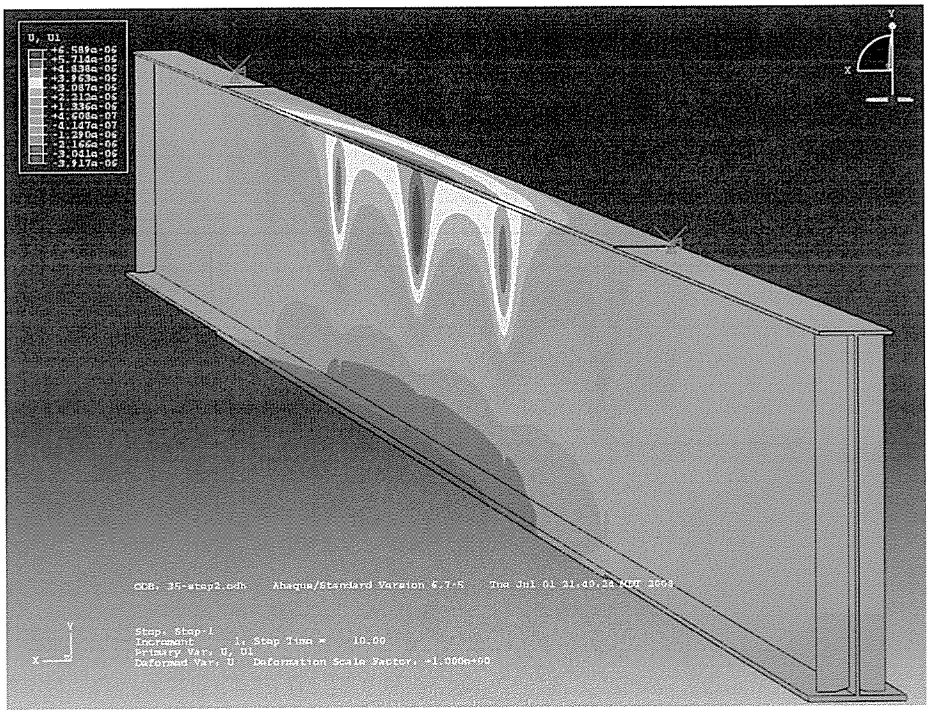

Figure 1 Lateral Displacement

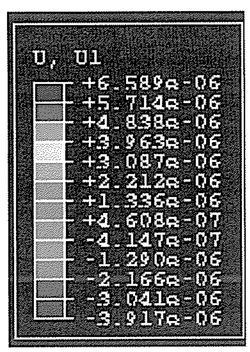

Figure 2 Displacement Values (ft.)

Results, such as these, were obtained with some models having two times the forces of gravity acting on them. Lateral forces were applied, end rotational displacements were tried, and laterally offsetting the pick points was also tried, all to no avail. It was learned that a "RIKS" analysis is required to run LTB analysis. Still, in spite of many hours of attempting this method, it did not work either. In order to

progress onward with FEA of LTB, one would need access to at least one of three things:

- Someone who knows how to do LTB analysis with the program,
- The appropriate manuals for the program version and analysis type, or
- Technical help from the analysis software development company.

In order to complete this project in a timely manner, the analysis was discontinued and a greater emphasis was placed on a literature search of related research.

3 Background and Previous Work

3.1 Lateral-Torsional Buckling of Tapered I-Beams (1981)

Thomas G. Brown (1981) considers critical loads acting at different heights with respect to the centroid on simply supported and cantilevered beams. Top flange, centroid, and bottom flange loadings are considered for simply supported and cantilevered beams. Tapered and prismatic, simply supported beams, were analyzed with a centrally concentrated and uniform loads. Tapered and prismatic cantilevered beams were analyzed with only a concentrated load at the tip. The simply supported beam is tapered to have increased depth at the mid span, while the cantilevered beam tapers such that the beam is deepest at the support.

The beam size used for the study was an idealized I-section with a maximum depth of 24.0 in. The beam had a flange width of 6.0 in., flange thickness of 0.5 in., and a web thickness of .375 in.

Through mathematical derivation, eigenvalues were extracted and coefficients γ were determined. In Table 1 through Table 4, γ is a dimensionless load, and the ratio of $\gamma / \tilde{\gamma}$, is a reduction factor from the respective prismatic case. Taper ratio, α, is inverse for the simply supported beam and the cantilevered beam. For the simply supported case, α is a ratio of the beam depth at the supports to the beam depth at the mid span. For the cantilevered case, α is the beam depth at the tip to the beam depth at the support. In either case, as α approaches one, the beam becomes prismatic; the lower α is, the greater the taper.

Table 1 shows the values determined with the central concentrated load applied at the top, center, and bottom for the simply supported beam.

Table 1 Coefficients for Simply-Supported Beams under Central Concentrated Loads

Taper ratio	Top Flange Load		Centroidal Load		Bottom Flange load		Capacity reduction
α	γ_1	$\gamma_1/\tilde{\gamma}_1$	γ_1	$\gamma_1/\tilde{\gamma}_1$	γ_1	$\gamma_1/\tilde{\gamma}_1$	at top
0.167	11.96	0.727	19.80	0.783	31.13	0.840	38%
0.333	13.02	0.792	21.13	0.835	32.58	0.879	40%
0.500	13.96	0.849	22.30	0.882	33.82	0.913	41%
0.667	14.85	0.903	23.37	0.924	34.96	0.944	42%
0.833	15.66	0.952	24.36	0.963	36.01	0.972	43%
1.00	16.44($\tilde{\gamma}_1$)	1.00	25.28($\tilde{\gamma}_1$)	1.00	37.05($\tilde{\gamma}_1$)	1.00	44%

The results presented in Table 1 show that the capacity of the beam when loaded at the top flange increases with a decreasing taper. Similarly, the beam capacity, when

loaded at the bottom flange, also increases with a decreasing taper. For a given taper, the beam capacity increases as the load position moves downward. As the taper decreases the difference between the top and bottom loading capacity becomes less significant.

Table 2 shows the values determined with a distributed load on a simply supported beam.

Table 2 Coefficients for Simply-Supported Beams Under Uniform Load

Taper ratio α	γ_1	$\gamma_1/\tilde{\gamma}_1$
0.167	22.93	0.778
0.333	24.59	0.834
0.500	26.01	0.882
0.667	27.34	0.927
0.833	28.34	0.964
1.000	29.49 ($\tilde{\gamma}_2$)	1.000

The results show that as the beam taper decreases the beam capacity increases. The effects of load position for a distributed load were not presented, presumably because they would be similar to that of the point load. Furthermore, distributed loads are commonly applied along the top flange, and in a few cases, along the bottom flange.

These results show that the location of load application is critical to the determination of LTB capacity of the beam. Table 3 summarizes the γ_1 values for a cantilevered beam with a concentrated load at the tip.

Table 3 Coefficients for Cantilever Beams With End Concentrated Loads

Taper ratio α	Top Flange Load γ_1	$\gamma_1/\tilde{\gamma}_1$	Centroidal Load γ_1	$\gamma_1/\tilde{\gamma}_1$	Capacity reduction at top
0.167	5.32	2.59	7.10	0.89	75%
0.333	4.03	1.96	7.29	0.91	55%
0.500	3.18	1.55	7.47	0.94	43%
0.667	2.65	1.29	7.65	0.96	35%
0.933	2.35	1.14	7.81	0.98	30%
1.000	2.06 ($\tilde{\gamma}_1$)	1.00	7.98 ($\tilde{\gamma}_1$)	1.00	26%

There are interesting differences in the outcome between simply supported and cantilevered beams. With a top flange loaded cantilevered beam, the capacity decreases with a decrease in taper, whereas, when loaded at the centroid, the beam capacity increases with a decrease in taper. For a given taper, beam capacity increases as the load position moves down. As taper decreases a greater difference exists between the top and bottom loading capacity. The particular effect of differing capacities between different degrees of taper is more pronounced in the cantilevered beam.

A prismatic top loaded cantilever beam has only 26% of the capacity of the prismatic centroid loaded beam. Table 4 illustrates this change in capacity of a prismatic cantilevered beam with the load at different heights, 0.0 being loaded at the centroid and 0.5 being loaded at the top.

Table 4 Coefficients for Prismatic Cantilevers with End Concentrated Loads

Load location a/d	γ_1
0.000	7.770
0.083	6.240
0.167	4.840
0.250	3.680
0.333	2.920
0.417	2.430
0.500	2.060

Critical loads can then be calculated using either equation 1 or 2.

$$P_{cr} = \gamma_1 \frac{\sqrt{EI_{yo}GK_0}}{L^2} \qquad \textbf{Equation 1}$$

and

$$q_{cr} = \gamma_2 \frac{\sqrt{EI_{yo}GK_0}}{L^3} \qquad \textbf{Equation 2}$$

where

E = Modulus of Elasticity

I_{y0} = Moment of inertia about the weak axis at the deepest section

G = shear modulus

K_0 = St. Verant torsion constant

L = span

The second beam scenario, see Table 2 on page 6, of this article presents great similarities to the girder in consideration in this project.

3.2 Lateral-Torsional Buckling Of Non-prismatic I-Beams (1996)

Gupta et al. (1996) presented a finite-element formulation to determine the LTB loads of continuous, non-prismatic I-beams. The main purpose of the article was to present the development of a FEA program capable of analyzing LTB. The computer program LTBAP (Lateral Torsional Buckling Analysis Program) was developed to determine the lateral stability of non-prismatic I-beams. Much testing was conducted to assess the accuracy of the program, and it has been used to successfully analyze beams with stepped, linear, and quadratic taper with reasonable accuracy. It also predicts, with reasonable accuracy all three cases of taper in the web, flange width, and flange thickness of I-beams. The mathematics behind the program development were shown.

Of relevance to this project, the conclusion is reached in the article, that a simply supported tapered beam, that has a higher moment of inertia at the middle of the beam, has a significant increase in the buckling capacity. The girder of the Bitchcreek Bridge has increased moment of inertia in the middle portion of the girder.

3.3 Lateral Torsional Buckling Of Singly Symmetric I-Beams (1997)

Helwig et al. (1997) conducted a study to derive equations for the design specifications for lateral torsional buckling of singly symmetric I-beams with transverse load applied at different heights. Design specifications, at that time, were relevant to beams loaded at mid-height; however, for top flange loaded beams, or bottom flange loaded beams, the LTB capacity of the beam was not represented well by the specifications at that time.

The 2nd edition of the AISC Steel Construction Manual LRFD has moment gradient modification factor as:

$$C_b = \frac{12.5M_{\max}}{2.5M_{\max} + 3M_A + 4M_B + 3M_C}$$ **Equation 3**

To remedy the complexity added by the point of application of the load along the height of a girder, the authors proposed a correction factor, C_b^* from their analysis results:

$$C_b^* = 1.4^{2y/h} C_b$$ **Equation 4**

To adjust for double curvature, the following was proposed:

$$C_b = \left[\frac{12.5M_{\max}}{2.5M_{\max} + 3M_A + 4M_B + 3M_C}\right] R \le 3.0$$ **Equation 5**

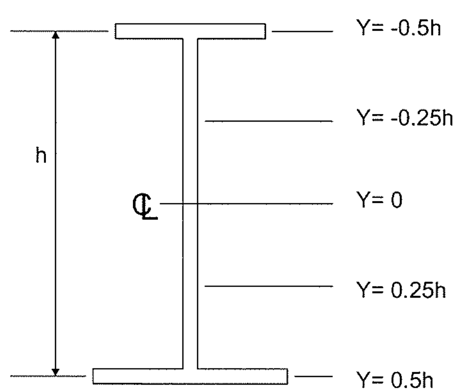

Figure 3 Load Height Definition

where:

y = vertical distance, as shown in Figure 3

h = depth of the girder, as shown in Figure 3

M_{max} = Absolute value of the maximum moment applied to the unbraced length of the beam

M_A = absolute value of the moment at the quarter point

M_B = absolute value of the moment at the mid point

M_C = absolute value of the moment at the three-quarter point

R = Cross section monosymmetry parameter

For beams subjected to single curvature bending, R=1.0. For cases of reverse-curvature bending:

$$R = \left(0.5 + \left(\frac{I_{yc}}{I_y} \right)^2_{Top} \right) \qquad \textbf{Equation 6}$$

where:

I_{yc} = Moment of inertia of the compression flange about the y-axis.

I_y = Moment of inertia about the y-axis.

The 13th Edition of the AISC Manual contains equations 5 and 6, just as proposed by the authors.

3.4 Lateral Torsional Buckling of Stepped Beams (2003)

Park and Stallings (2003) studied and presented this article on stepped I girders of continuous multi-span bridges. Near the interior supports of a multi-span bridge with continuous girders, the girders often have increased cross sections to accommodate the high negative moment. End spans typically have a single cross section step with one end of the girder having a greater torsional resistance to LTB than the other end. The middle spans would have two potentially symmetric cross section steps with the ends having a greater cross sectional area, and being more rigid than that at the center of the span. These cross section steps make calculating the LTB capacity of the beam more difficult. The 1998 American Institute of Steel Construction LRFD Specifications only account for prismatic and web-tapered beams. The authors propose a mathematical formula to account for these steps in determining the LTB capacity.

The 1998 edition of AISC LRFD Specifications define the LTB critical moment capacity as:

$$M_{cr} = C_b \frac{\pi}{L_b}\sqrt{EI_y GJ + \left(\frac{\pi E}{L_b}\right)^2 I_y C_w} \qquad \textbf{Equation 7}$$

In which

$$C_b = \frac{12.5M_{\max}}{2.5M_{\max} + 3M_A + 4M_B + 3M_C}$$ **Equation 8**

where:

C_b = moment gradient modifier

L_b = laterally unbraced length

E = modulus of elasticity of steel

G = shear modulus of elasticity of steel

J = St Venant torsional constant of the cross section

I_y = moment of inertia of the cross section about the weak axis

C_w = warping constant of the cross section

Equations for C_b, such as the one presented above, have been shown to have insufficient accuracy for stepped beams. To calculate the critical LTB moment under uniform bending, the authors proposed the following equation:

$$M_{ost} = C_{st} M_{0cr}$$ **Equation 9**

in which

$$C_{st} = C_0 + 6\alpha^2\left(\beta\gamma^{1.3} - 1\right)$$ **Equation 10**

for doubly stepped beams and

$$C_{st} = C_0 + 1.5\alpha^{1.6}\left(\beta\gamma^{1.2} - 1\right)$$ **Equation 11**

for singly stepped beams where:

M_{ost} = critical LTB moment

M_{ocr} is found by calculating M_{cr} with $C_b = 1$ and other variables as a prismatic beam of the same length having the smaller cross section along the entire span

C_0 = constant equal to 1 for constant moment loading

$C_0 = 1$ for single zero moment point cases

$C_0 = 0.85$ for double zero moment points

α = ratio of length of the increased cross section on one end to the entire length

β = ratio of the increased flange width to the main flange width

γ = ratio of the increased flange thickness to the main flange thickness

To calculate the critical LTB moment under general loading, the authors proposed:

$$M_{st} = C_{bst} C_{st} M_{ocr}$$ **Equation 12**

with a new moment modification factor:

$$C_{bst} = \frac{10M_{\max}}{4M_{\max} + M_A + 7M_B + M_C}$$ **Equation 13**

This article, while very interesting and helpful for the case of girders tapered at the supports, it is not directly applicable to the girder of this project. However, it could come to be of great worth when the opportunity arises to place girders of this nature.

4 What Happened Out There?

These studies conducted helped me to understand what happened, why the first scenario didn't work, what could have been changed to make it work, and why the second scenario worked.

4.1 Why the First Attempt Didn't Work

There are multiple factors that contributed to the complexity of the problem as to why the initial method of lifting the girder for the replacement bridge didn't work and the girder was going to buckle. To mathematically predict the failure, each of these factors would have to be addressed. The current literature, however, presents no more than two of these factors combined. The factors that are most likely to have contributed to the buckling of the replacement girder are listed below.

First, the unbraced length, approximately 187 ft, was simply too great. For the nature of the girder, considering length and the properties discussed below, the pick points were simply too far apart.

Secondly, the girder was curved. Due to the heating effects of welding the plates together during fabrication, the girder had a lateral bow, approximately 10 in. at the mid span. Although not a large bow in a nearly 200 ft girder, this would greatly exacerbate the tendency to laterally buckle. The girder was essentially already in a compression flange buckled state before the stresses of lifting was applied. No research has been found on what effect lateral bow would have on the LTB capacity of plate girders. There was obviously some capacity remaining, but the value of that capacity was unknown.

Third, this girder was singly symmetric. When girders are designed, they are designed to withstand the forces of the completed structure. When the bridge was finished the girder became composite with the reinforced concrete decking. In design, the composite deck is considered as acting in unison with the top flange; thus the required size for the top flange decreases. This is why composite plate girders commonly have smaller top flanges then bottom flanges, posing a problem to erectors. The top flange often lacks strength that would lend to more economical construction methods. In terms of cost, however, engineers must take into account the other effects. If the replacement girders were doubly symmetric, the girders would have weighed approximately 16,000 pounds more. The current cost of fabricated steel is approximately $1.75 per pound. The added material would have then cost an extra $28,000 per girder; and an extra $112,000 for the whole bridge. The extra weight would need to be taken into account during analysis. Questions such as: a) would the girder be self stable during erection, and b)

would the added cost of a larger crane to pick the heavier section offset the savings of, an otherwise, more economical method, would need to be answered.

Fourth, the girder, like many girders typically do, had a camber built into it (Figure 4). The upward camber is build into girders such that upon being loaded with the weight of the deck, and everything else, the girder would be flat. The camber raised the center of gravity of the girder; and the greater the camber the higher the center of gravity. The problem is that the higher the center of gravity is, the closer the pick points, at the ends, are to the center of gravity, and the laterally weaker the girder becomes.

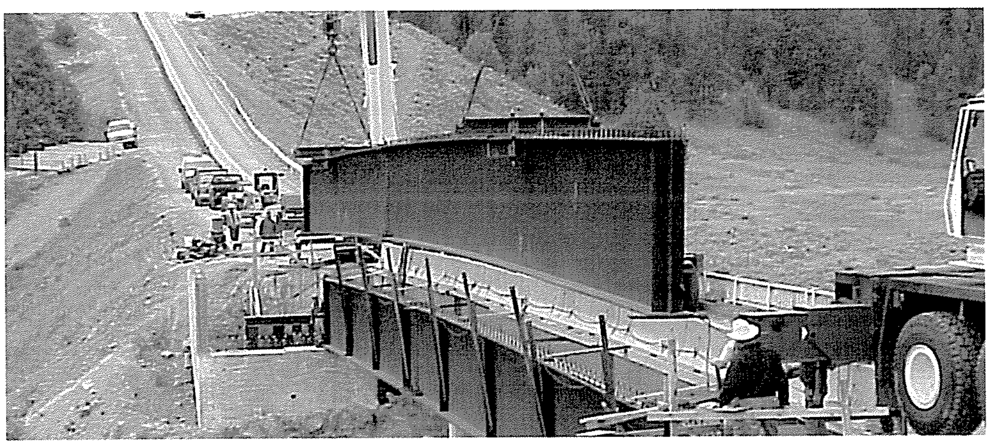

Figure 4 Camber In a Girder

Fifth, the 133 ft stiffening truss was centered over the 187 ft. unbraced length, leaving about 27 ft of unstiffened girder on each end of the unbraced length. The truss was centered on the increased cross section of the middle portion of the girder. The truss overlapped the weaker portion of the girder by approximately 7 ft. on each end. Essentially the truss was only over the stronger portion of the girder. The weaker ends have a top flange moment of inertia, about the girder's weak axis, of only 5.20% of the moment of inertia of the thicker flange in the center of the girder plus the added moment of inertia of the stiffening truss.

Sixth, the truss to girder connection method was not adequate. The connection had some sloppiness which gave the girder some wiggle room to move laterally with respect to the truss. The "wiggle space" let the girder bow even further before the truss engaged and began to offer the intended support. To run a simple rough calculation of the capacity, assume hypothetically, the truss had been rigidly connected the full unbraced length of the girder, and the girder was prismatic with the smaller top flange the entire length. The combined top half of the girder and the truss were in compression. The buckling capacity of this top-half unit about the weak axis of the girder would be 535 kip. The maximum moment at mid span of the actual girder and truss was 2190 kip-ft. This equates to 303 kip of compression, which is approximately 57% of the lateral buckling capacity of the girders. Based on this simple calculation one could conclude that a full length rigidly connected truss would probably have been sufficient to prevent buckling of the girder.

Seventh, the girder was lifted using 2 "girder dogs" (see Figure 5) that allowed the girder to rotate longitudinally about a pivot point at the top flange, approximately 3 ½ ft above the center of gravity. The definition of unbraced length, L_b, given by the thirteenth edition of the AISC Manual is the "length between points that are either braced against lateral displacement of compression flange or braced against twist of the cross section". In this situation, the only braced characteristic about the scenario was that the center of gravity was below the longitudinal pivot point of the lifting arrangement. Also, to define the unbraced length, one must use the length between these pick points, at which, there is only a *resistance* to overturning. For a nearly 200ft girder, there certainly wasn't much resistance to twisting of the cross section. The closer the longitudinal pivot

point is to the center of gravity, the closer the moment capacity of the beam will approach the moment capacity of the girder about the weak axis. With the pivot point at the center of gravity, the girder would freely rotate to the side and the problem would turn into a simple issue of bending about the weak axis.

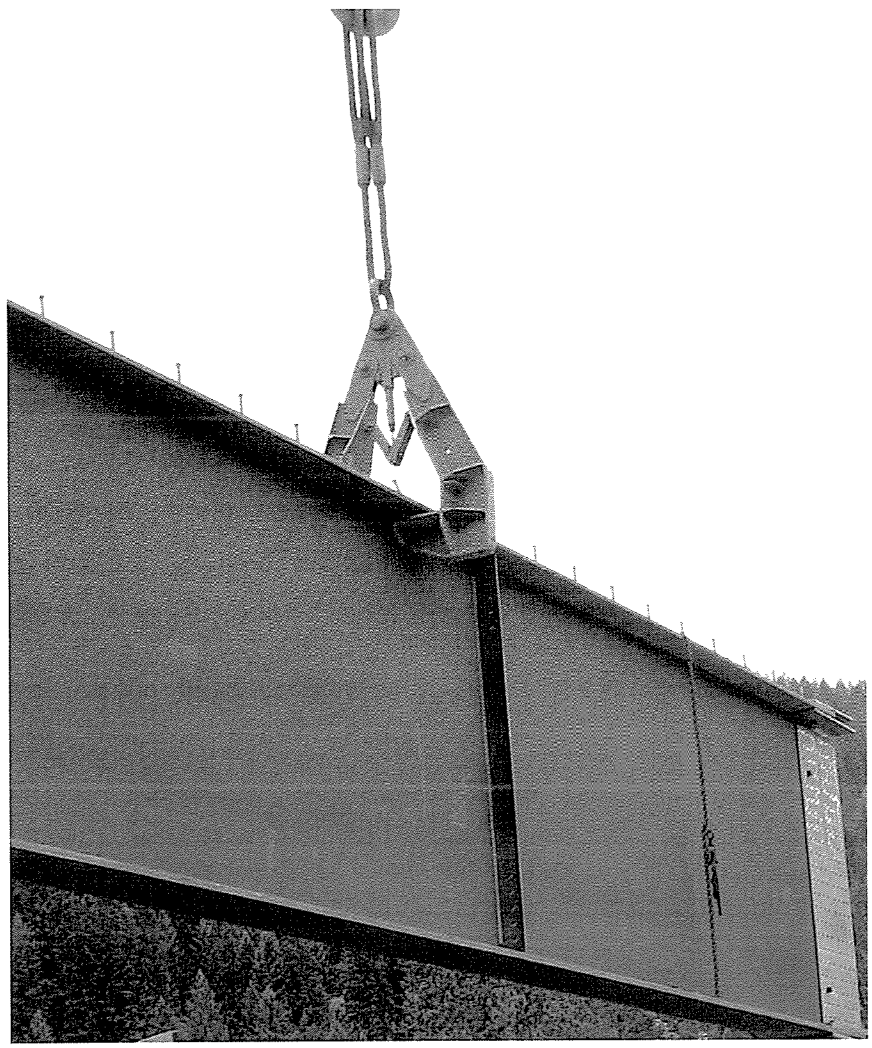

Figure 5 "Girder Dog"

Eighth, the stiffening truss placed a weight on the top of the flange. The capacity of a top loaded beam (Brown, 1981) is only 64 % of the same beam loaded at the centroid

(see Table 1). This effect was very small in this case because the truss only weighed approximately 2000 lb., while the girder weighed 96,700 lb. The truss raised the center of gravity about one inch. This effect, for practical purposes, would probably be negligible.

4.2 Why the Second Attempt Worked

The second phase, or the second method for picking the girder, had many of the same inherent problems, but there were two additional factors that were different. These would also need to be included in any analysis. The girder still had the bow, the camber, and the singly symmetric design. The second time, however, the length between pick points was reduced by 35% to approximately 122 ft.

The stiffening truss was removed, but its effect wouldn't have been missed because it was not present at the beginning of the operation due to the "sloppy" connections.

Because the support points were moved inward to increase the LTB capacity, this left 35 feet of girder cantilevered out on each end. The end portions of the girder would have induced double curvature. The AISC manual does not have provisions to account for such effects. Currently the AISC manual instructs the use of the full unbraced length in the LTB calculations; however, Helwig et al. proposed equation 6 (page 12), that would reduce the C_b factor for double curvature.

Next to reducing the unbraced length, perhaps the best thing that helped stabilize this girder was the use of a "hitch", shown in Figure 6, that securely fastened to the top flange of the girder and raised the longitudinal pivot point far above the girder. This would help stabilize the girder by offering much greater resistance to overturning.

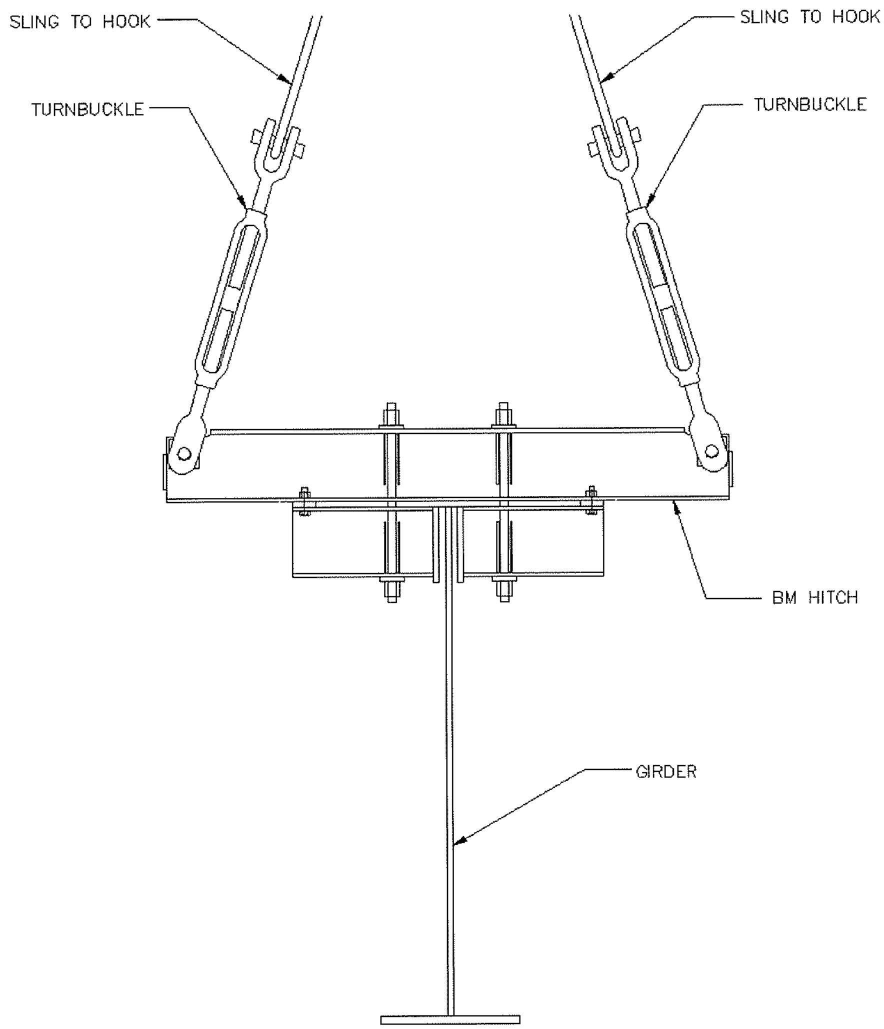

Figure 6 The Hitch Used in the Second Attempt

5 Conclusion

The Bitchcreek girder, as it is in the "real world", had many characteristics that thwarts any attempt to accurately predict failure by hand computation. There are simply too many factors that can not be accurately predicted. Several of the effects created by the characteristics of this girder have not yet been researched, or could only be done with finite element analysis, such as the lateral curve, vertical camber, non-prismatic flanges, "sloppy" truss connection, and the variation of the height of the longitudinal pivot point. This situation is one that would have greatly benefited by the use of finite element.

From this project, it has been learned that to accomplish the first scenario, a full length truss rigidly connected would be of much benefit. Raising the height of the longitudinal pivot point has some good effect. Overall, the only way to truly know what the girder will do is to abandon any attempt at hand calculations and resort to a finite element analysis of the girder. It is the only way, short of a live test, of knowing what will happen.

To successfully run an analysis, it was learned that an engineer must have either the knowledgeable personnel, appropriate manuals, or program technical help in order to be properly trained and learn the program.

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